MATHS CLUB STARTER PACK

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How to use this pack

The pack contains a series of activities. Go through the pack and pick and activity to use for your maths club. You don't have to do the activities in order. An activity may take more or less than one maths club session.

The first half of the pack contains the student notes. These should be shown to all students in the club. They describe the activity and give questions.

The second half of the pack contains the facilitator notes. One person, either the teacher or a lead student should look at the facilitator notes. These are not for all the students to see.

The facilitator notes are notes on how to deliver the activity and also contains the answers. The facilitator **should not** give the answer to the class, even if they don't get it. The facilitator may give hints and help so students can get the answers on their own.

The facilitator notes are designed so that a lead student can understand them. The idea is that activities can take place, even when the patron is busy.

Please complete the feedback page after each activity to help us improve the pack.

Contact Zach Mbasu: +254 722124199, zmbasu@yahoo.com or admin@africanmathsinitiative.net if you have any further questions about using this pack.

Enjoy!

Contents

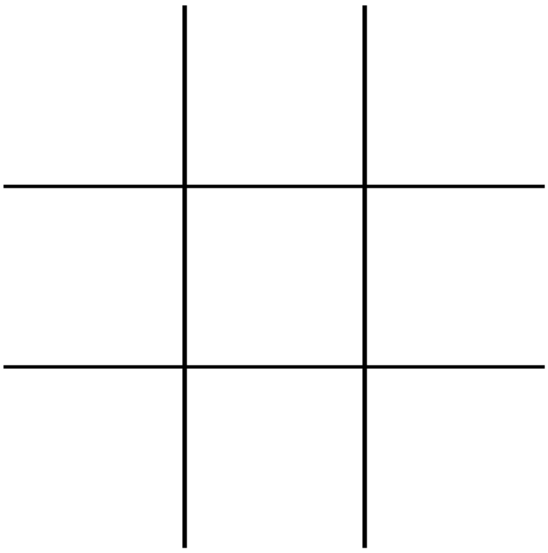
Date	Activities Used	Feedback and Comments Did you understand the problem? Did you enjoy the problem? What was the biggest challenge?	Our Rating /10

Student Versions

Games

Tic Tac Toe/O-X

Draw the grid below to make a grid with 9 spaces.



This is a 2 player game.

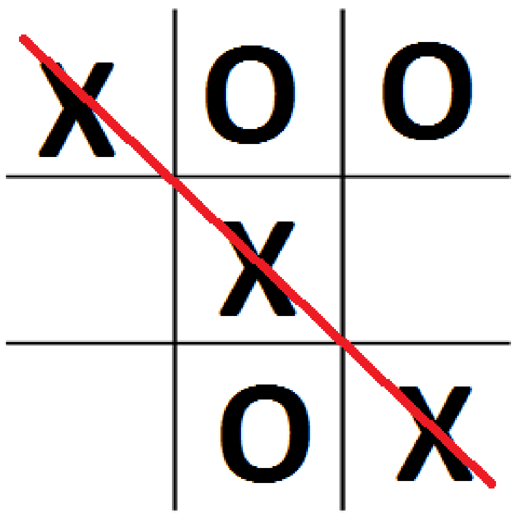
One person has the **O** symbols the other has **X**.

Take it in turns to put one of your symbols in an empty space.

The winner is the first person to get 3 of their symbols in a line (horizontal, vertical or diagonal).

Example

Player **X** is the winner because they have 3 **X**s in diagonal line.



15 Game

Only play this game after you have played Tic-tac-toe!

Write the numbers 1 to 9 in a line on paper.

1 2 3 4 5 6 7 8 9

This is a 2 player game. Take it in turn to pick one number at a time.

The winner is the first person to have exactly three numbers that add up to 15. **Example**

1 2 3 $\boxed{4}$ $\boxed{5}$ $\boxed{6}$ 7 8 9

The player using squares has won because 4 + 5 + 6 = 15. The player using circles has 7 and 8 which makes 15 but you only win if you have **exactly 3** numbers that make 15.

After playing many times now write the numbers in a square like this and play the same game again.

8	1	6
3	5	7
4	9	2

What do you notice about playing the game this way?

Do it remind you of anything you have done before?

Patience

This is a relay race requiring strategy to win. You will need a set of cards for each team with the numbers 1 through 10 written on them. You can use playing cards Ace through 10 or you can create your own cards. You will also need a long table on which to place a set of cards for each team.

Divide the class up into teams of about ten students. Each team should have a monitor who will randomly spread out that team's set of cards face side down on the table and ensure that the rules are followed.

Team members line up and the first member of each team will run over to the table from a set distance and turn over a card. If the card turned over is a one, the card is left face up and the next student on that team tries to turn over a two. If the one is not turned over by the first student, the card is turned back over and the next student has to try to turn over a one and so on until a one is uncovered.

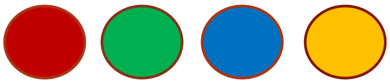
As it is a relay race, each student who turns over a card has to return to his team and tap the next person before that person can proceed. The goal is to be the first team to turn over all the cards in order.

Mastermind

Rules

2 players

Player 1 creates a 4-digit code using red, green, blue and yellow (e.g. r,g,b,y or r, r, g, y or y, g, b, g)



Player 2 attempts to guess the code in the fewest tries.

Each time player 2 guesses, player 1 will tell them how many colours they guessed in the correct position, and how many other colours they have guessed correctly but in the wrong position.

(see facilitator notes for full instructions and example game)

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Nim/21

Rules

Two player game.

One person starts counting from 1 and can stop at 1, 2 or 3.

The next player carries on counting from where the first player stopped and can say up to 3 more numbers.

The player who says 21 is the **loser**.

Example game:

Player 1	**Player 2**
1, 2	3, 4, 5
6, 7, 8	9
10, 11, 12	13, 14
15, 16, 17	18, 19, 20
21	**Player 2 wins!**

Nim-Related Games

NIM

Standard NIM is played with three piles of any number of objects (use coins, toothpicks, matchsticks, bottlecaps, whatever). Players take turns choosing **one** pile and removing any number of coins away from that pile. The person who takes the last object loses.

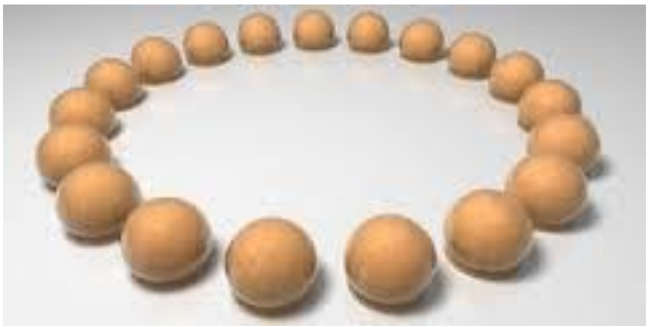


Subtraction NIM

You can take a maximum of k objects from a pile on each turn, where k is established by the players in advance. The person who takes the last object loses.

Circular NIM

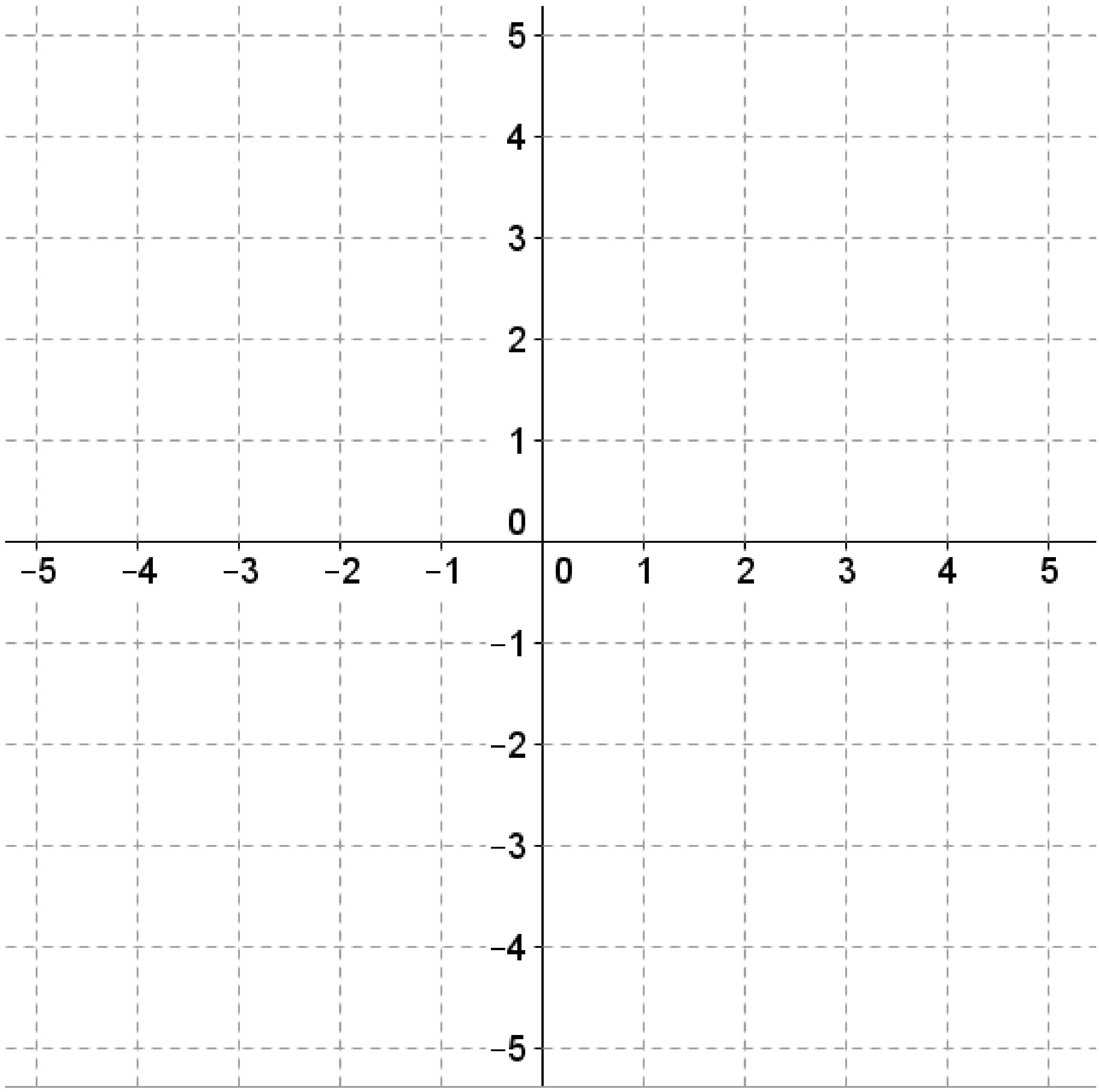
Place a number of objects in a circle. You can remove up to 3 adjacent objects on a turn. The person who takes the last object loses.



Making Squares

This is a game for 2 players (or 2 teams)

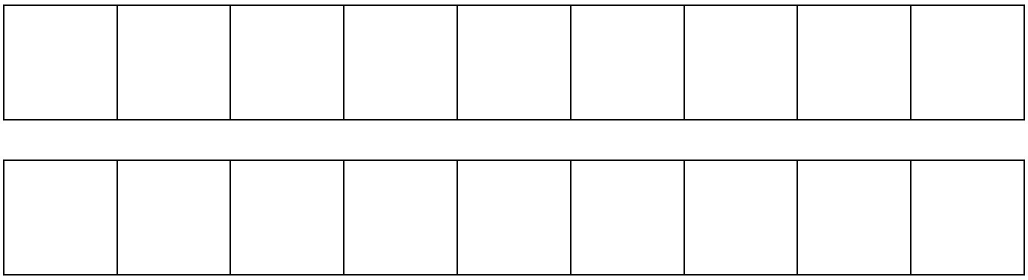
Take it in turns to plot a point on the coordinate grid for your team.



The first team to make a square using 4 of their points wins.

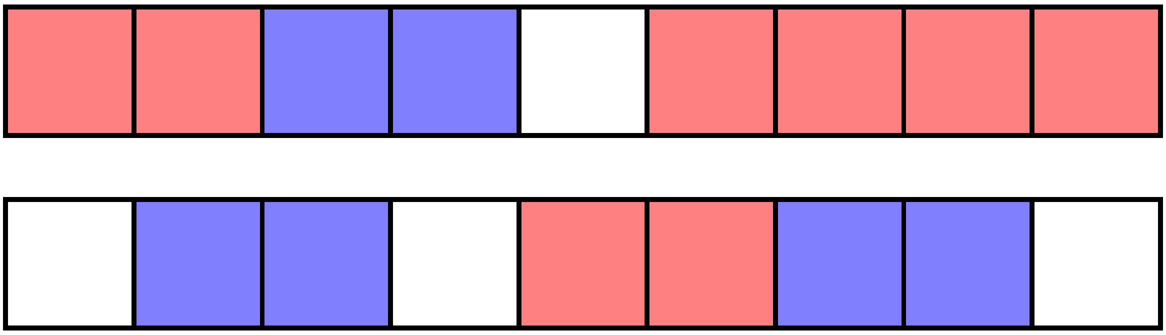
Paper Pieces

Two player game, played on two rows of squares (any length)



Players take turns to colour in **any two adjacent** squares. The first player who can’t make such a move is the **loser**.

Example: After Red's 4th turn, Blue cannot make another move and loses.

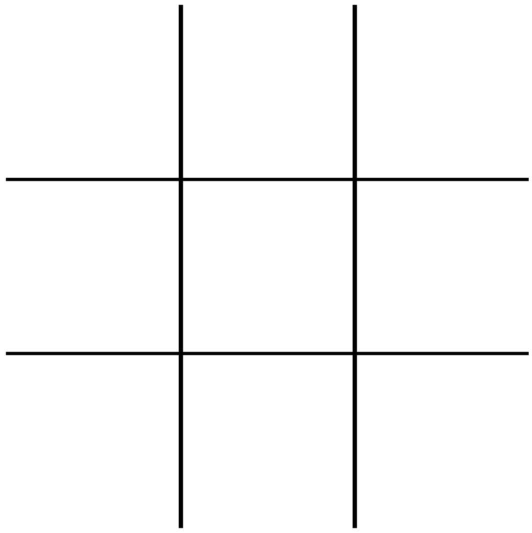


Variation: Play the game with just one row of squares.

Tic Tac Toe with levels

Draw the grid shown to make a grid with 9 spaces.

This is a 2 player game.

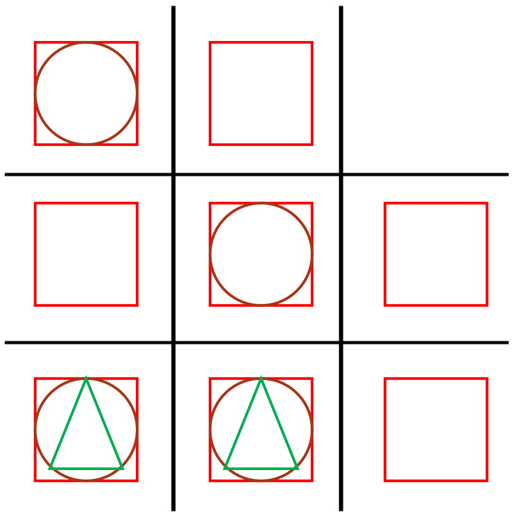


Take it in turns to either:

- draw a square in an empty space
- draw a circle inside a square
- draw a triangle inside a circle

The winner is the first person to complete a line of 3 spaces with the same symbol.

Example



In this game, the next player can win by putting a circle in the bottom right to make a diagonal line.

(It doesn't matter who draw the other symbols, only who completes it.)

Letter Matching

Only play this game after you have played Tic-tac-toe!

Write the following words in your book or on paper:

eat	book	less	Bits	lot	air	bee	lip	soda

This is a 2 player game. Take it in turn to pick one word from the list.

The winner is the first person to have exactly three words that contain the same letter.

Example

Player 1	Player 2
Eat	Less
Bits	Bee
Air	Soda
Lip	

Player 1 has won because **bits**, **air** and **lip** all contain the letter **i**.

After playing many times now write the numbers in a square like this and play the same game again.

eat	**bee**	**less**
air	**bits**	**lip**
soda	**book**	**lot**

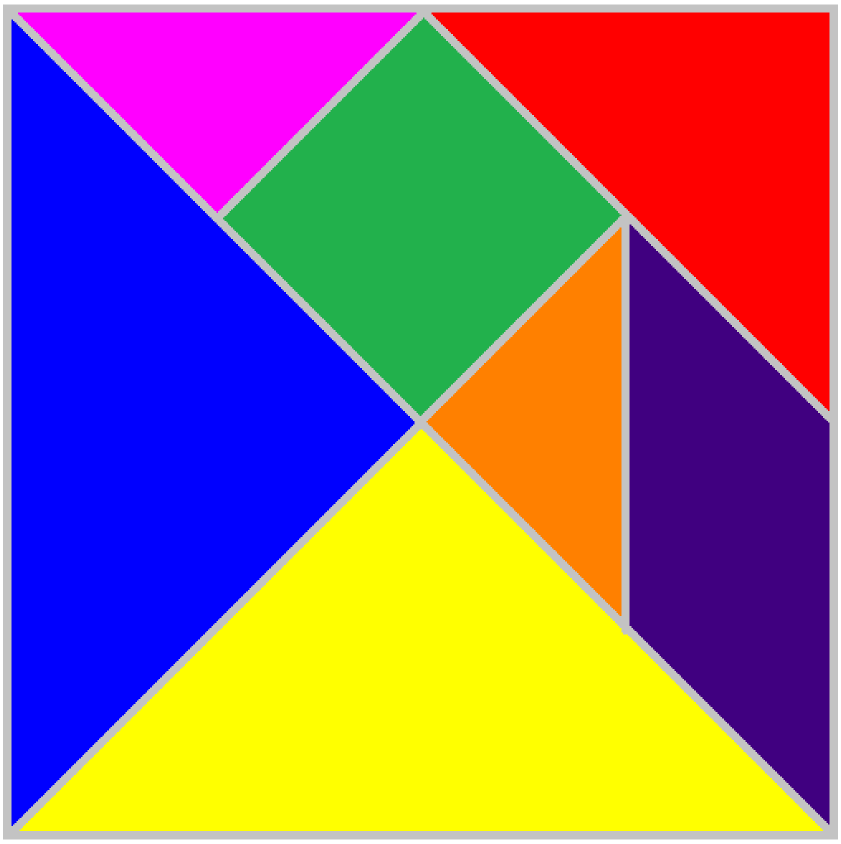
What do you notice about playing the game this way?

Does it remind you of anything you have done before?

Tangrams

You will need pen, paper and either scissors or a ruler for this game

Create a tangram by tracing the image below onto a square of paper, or try to construct using paper folding



Cut into individual pieces and see what shapes you can make

Can you make the square again? How about any animals? Or letters in the alphabet?

Countdown

This game can be played by any number of people

Pick any 5 small cards and 1 large

2	7	3	7	4	8	10	5	9	4	25	100
8	1	10	3	6	9	6	1	2	5	75	50

These are the game cards.

2	10	4	6	9	50	**Game cards**

Pick any other 3 small numbers. These combine to make the target (341)

3	4	1	**Target**

Everybody now has 30 seconds to see who can get closes to the target number using the game cards at most once each, and operations **+ - x ÷**

$$50 \times 2 \times 3 = 300$$

$$10 \times 4 = 40 \quad +$$

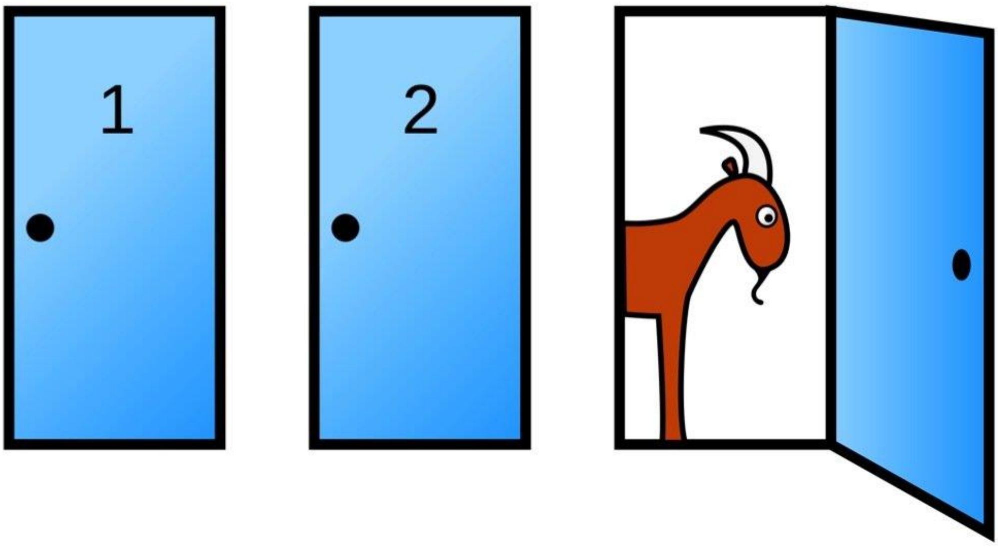
<u>340</u> Close but not exact!

Problems & Puzzles

Monty Hall Problem

#1 The Monty Hall Problem

Let's say you're on a game show where the host shows you three doors. Behind one of these doors is a brand new car. Behind the other two are goats. You get to pick a door. Then, the host will open one of the doors you didn't open to reveal one of the goats.



He asks you:

Do you want to switch doors?

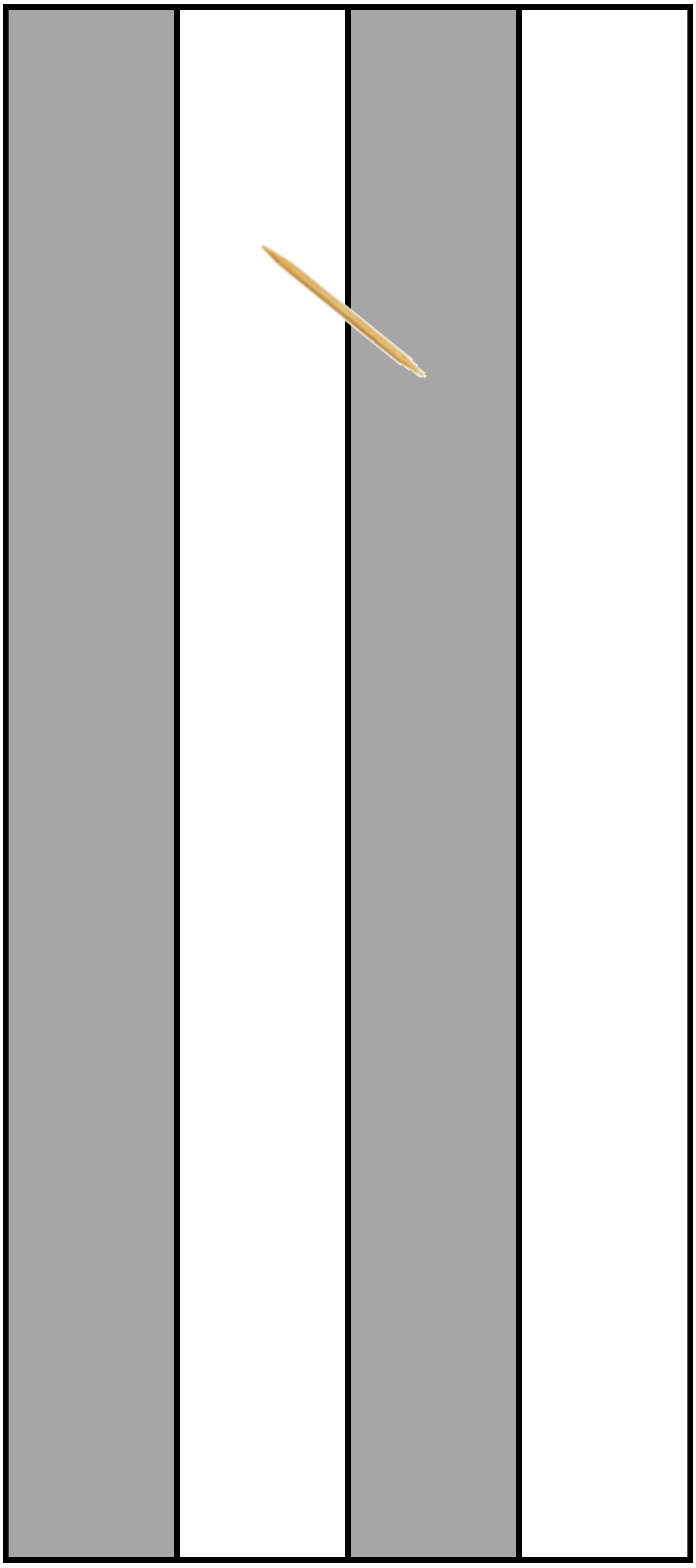
Or do you want to stay with the door you chose?

What do you do?

(see a full explanation of the problem in the facilitator notes)

Buffon’s Needle

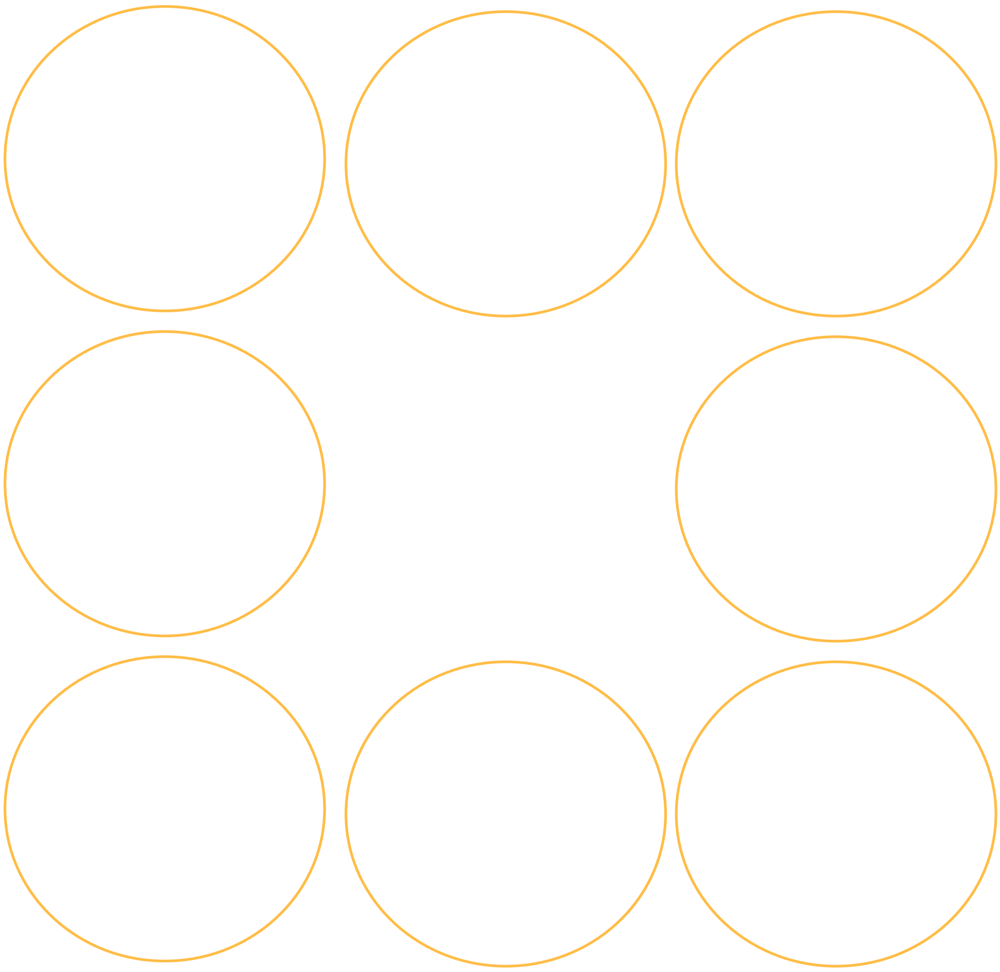
If you drop a small toothpick on the grid below, what do you think is the probability that it will cross over a line?



(See the facilitator notes for details of how to conduct an experiment and test your predictions)

15

Write the numbers 1 to 8 in the circles so that all of the rows and all of the columns add to make 15



=15

=15

=1
5

=1
5

Addition Square

There is something strange about this addition square. Can you find the missing number?

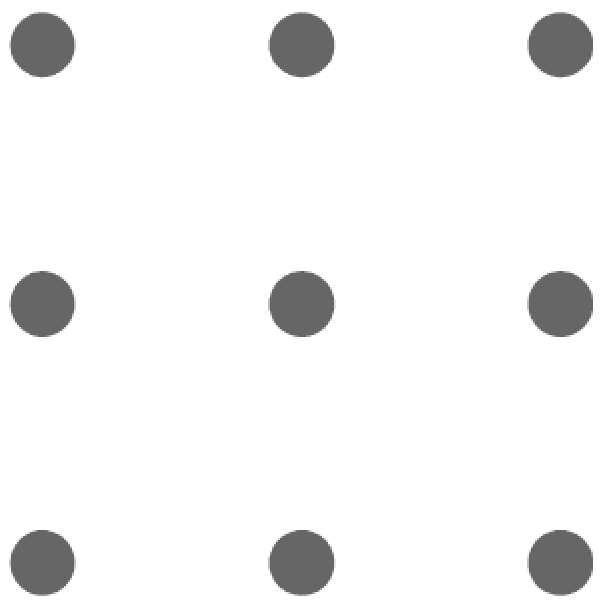
+	3	8	11
3	6	11	2
8	11	4	7
11	2	7	?

How about this square? It follows a different rule.

+	1	3	5
1	2	4	6
3	4	6	1
5	6	1	?

9 Dots

Go through all 9 dotes using only four straight lines without taking your pen off the page.



You may not go back over a line you have already drawn

(Do not look at the facilitator notes until you are sure you have the solution.)

Find the path

By moving up, down, left or right, make a path from the Start to the Finish.

Add up all the numbers you pass through.

	4	9	7	7	**4**	⇨ Finish
	8	9	4	5	7	
	6	6	4	9	9	
	7	8	8	8	6	
Start ⇨	**5**	5	6	5	5	

Can you make exactly 53?

What is the smallest possible number?

Can you make exactly 60?

Consecutive Sums

Some numbers are the sums of consecutive numbers, e.g.

$10 = 1 + 2 + 3 + 4$

$11 = 5 + 6$

$9 = 4 + 5$ and $2 + 3 + 4$

Can all numbers be written this way?

Which numbers can be written in more than one way?

IQ Challenge I

1. Fill in the missing numbers for each sequence

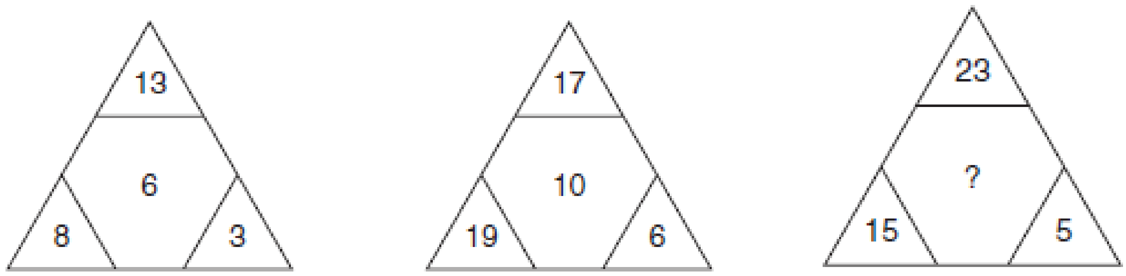
   0, 1, 4, 9, 16, 25, 36, 49, ?

   0, 100, 6, 94, 12, 88, 18, 82, ?, ?

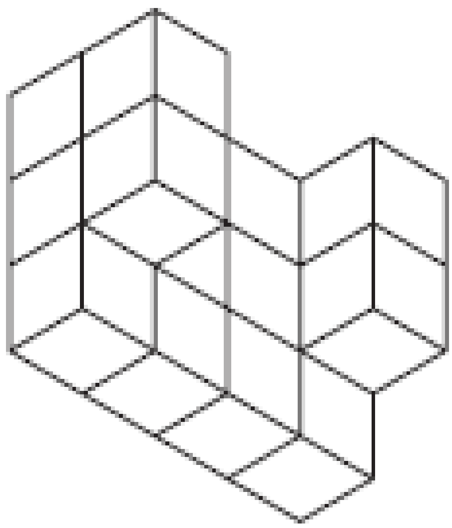
   1.5, 3, 5.5, 9, 13.5, ?

   5, 26, 131, 656, ?

2. What number should represent the ?

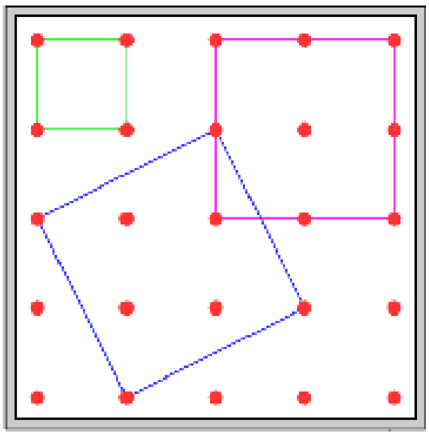


3. How many faces are visible on the shape below if you could look at it from all directions?



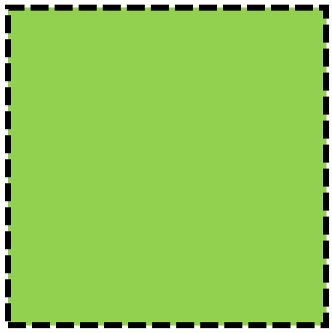
Counting Squares

How many different squares can you make by joining dots on a 5x5 grid?

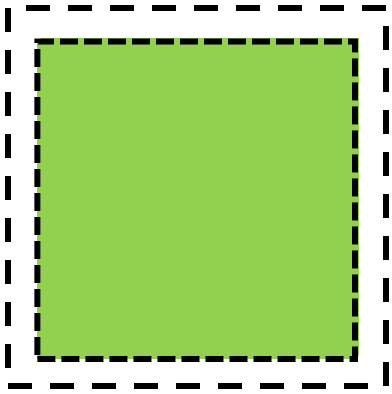


Fence Around a Field

A fence of exact length is build around a square field



If there was an additional 1m of fencing, how much gap would there be between the field and the fence?



(Assume equal space is left around all sides)

Is it enough space for a mouse to fit through? How about a tractor?

Number Challenge I

1. John thinks of three numbers. He picks two of them and adds them to get 11. He does this again and gets 17 and again to get 22.

What are the three numbers?

Is there a method to do this for any three totals?

2. Write the numbers 1, 1, 2, 2, 3, 3 in a line such that there is one number between the 1s, two numbers between the 2s and three numbers between the 3s.

Example:

1	3	1	2	3	2

One number between the 1s, but only one number between the 2s and only two numbers between the 3s. So this is wrong!

Harder: add two 4s and make sure there are four numbers between the 4s.

3. Write an equation that only uses the digits 9,9,9,9 and equals 100.

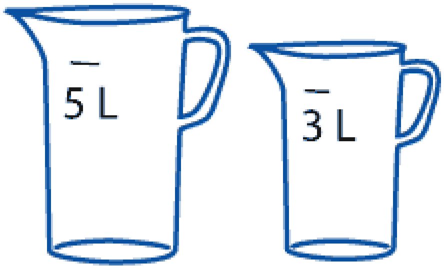
9 x 9 + 99

Uses four 9s but doesn't equal 100!

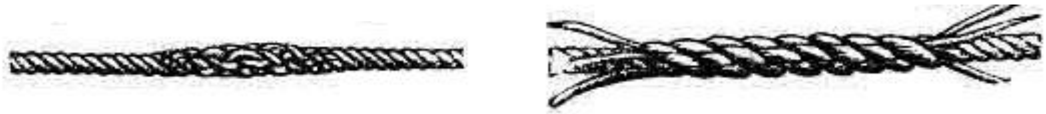
Try to find a correct equation!

Logic Puzzles I

1. A man has two empty jugs: a 3 litre jug and a 5 litre jug. The jugs have no markings on them. How can he measure exactly 1 litre of milk without wasting any?



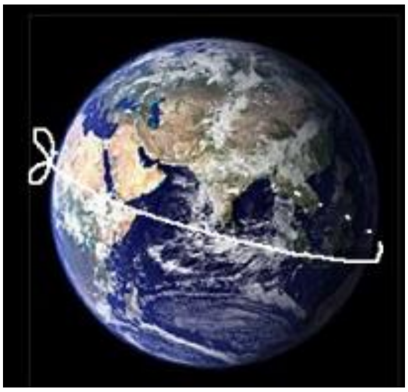
2. You have two ropes, each takes one hour to burn completely. Both of the ropes are not the same thickness all way through, meaning that some sections of the ropes will burn faster than others. Using the two ropes and matches, how could you time 45 minutes?



3. A woman goes to the river with two empty buckets: a 3 litre bucket and a 5 litre bucket. The buckets have no markings on them. How can she bring back exactly 7 litres of water?

Geometry Puzzles I

1. Imagine that the Earth is a perfect sphere and imagine we have a large piece of string tied around the equator.

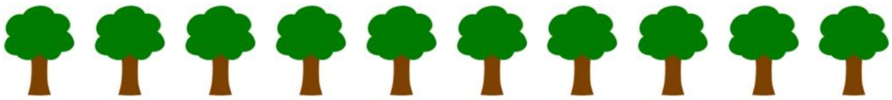


We then add one meter to the length of the string so that now the string is hovering above the equator, still in a circle.

What is the size of the gap between the equator and the piece of string? Enough to fit a piece of paper under? A cat? A car?

2. How can you cut a round cake into 8 equal pieces with only three straight slices of a knife.

3. A farmer wants to plant 10 trees in five rows, with four trees in each row. How can they do this?



Special Numbers

Can you guess my special two digit number?

My number is special because if I add the **sum** of its digits to the **product** of its digits, it gives my original number!

What could my number be?

For example, try 24.

The **sum** of the digits is 2 + 4 = 6

The **product** of the digits is 2 x 4 = 8

Since 6 + 8 is not 24, 24 is not a **special** number.

1. Can you find a special number?
2. Can you find more than one?
3. Can you find them all?

Poisoned Wine Puzzle

The King of a small country invites 1000 guests to his annual party. Each guest brings the King a bottle of wine. Soon after, the Queen discovers that one of the guests is trying to assassinate the King by giving him a bottle of poisoned wine. Unfortunately, they do not know which guest, nor which bottle of wine is poisoned. Just a single drop of the poisoned wine would kill the King.

The King has some prisoners in his dungeon. He decides to use them as taste testers to determine which bottle of wine contains the poison.

The poison, when taken, has no effect until exactly 24 hours later when the drinker suddenly dies. The King needs to determine which bottle of wine is poisoned by tomorrow so that the festivities can continue as planned. Hence he only has time for one round of testing. How can the King administer the wine to the fewest number of prisoners to ensure that 24 hours from now he is guaranteed to have found the poisoned wine bottle?

Magic Cards

CARD 0

01 03 05 07 09 11
13 15 17 19 21 23
25 27 29 31 33 35
37 39 41 43 45 47
49 51 53 55 57 59
61 63

CARD 1

02 03 06 07 10 11
14 15 18 19 22 23
26 27 30 31 34 35
38 39 42 43 45 47
50 51 54 55 58 59
62 63

CARD 2

04 05 06 07 12 13
14 15 20 21 22 23
28 29 30 31 36 37
38 39 44 45 46 47
52 53 54 55 60 61
62 63

CARD 3

08 09 10 11 12 13
14 15 24 25 26 27
28 29 30 31 40 41
42 43 44 45 46 47
56 57 58 59 60 61
62 63

CARD 4

16 17 18 19 20 21
22 23 24 25 26 27
28 29 30 31 48 49
50 51 52 53 54 55
56 57 58 59 60 61
62 63

CARD 5

32 33 34 35 36 37
38 39 40 41 42 43
44 45 46 47 48 49
50 51 52 53 54 55
56 57 58 59 60 61
62 63

The numbers on these cards have been chosen in a very special way so that you can be used for a magic trick. Can you spot anything special about them?

Russian Multiplication

Multiply together 22 and 19, writing down your answer. Then read the following instructions.

Write 22 on the left and 19 on the right. Multiply the left side by 2 every time and divide the right side by 2, and do this until the right side is equal to 1. If, while you do this, you ever get a decimal number on the right side, round down to the next integer.

22	19
44	9
~~88~~	~~4~~
~~176~~	~~2~~
352	1

The next step is to get rid of all the lines with an even number on the right side, which leaves us with 22, 44 and 352 on the left side. What do you notice when you add 22, 44 and 352? Now try this with other multiplications. Does it still work? If it does, can you explain why it works?

Langford’s Problem

Take the numbers 1,1,2,2,3,3. Write them in a line so that there is one number between the two ones, two numbers between the two twos, and three numbers between the two threes.

Take the numbers 1,1,2,2,3,3,4,4. Write them in a line so that there is one number between the two ones, two numbers between the two twos, three numbers between the two threes, and four numbers between the two fours.

Take the numbers 1,1,2,2,3,3,4,4,5,5. Write them in a line so that there is one number between the two ones, two numbers between the two twos, three numbers between the two threes, four numbers between the two fours and five numbers between the two fives.

One of the above challenges is impossible - can you work out which one and prove why it is impossible?

Coin game

In pairs play the following game a couple of times, and then answer the questions below.

One person is heads (player 1), the other person is tails (player 2).

Toss the coin a maximum of 10 times, player 1 gets a point if the coin lands on heads, and player 2 gets a point if the coin lands on tails. If one person becomes three points ahead of the other the game stops and they win. If no one is 3 points ahead after ten tosses the game is a tie.

Which of the following scores are impossible in this game and why?
5-2, 7-4, 6-2

Think of ways the game could end in a tie.

What is the chance of winning this game?
(use the table over the page to help)

Fill in the table shading out impossible final scores, highlighting winning final scores for Player 1 and Player 2, and highlighting final scores which indicate a draw.

Then write in each box how many ways there are to end up at each score.

A few boxes have been done for you – for example
3-0 is a winning score for player 1, and there is only 1 way to get to that score (1-0 then 2-0 then 3-0).

There are two ways to get to a score of 1-1 (0-1 then 1-1 or 1-0 then 1-1). There is a 1 in the 0-0 box as there is just one way to start the game

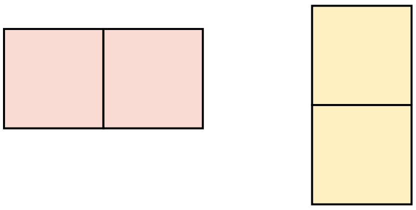
		Player 1							
		0	1	2	3	4	5	6	7
	0	1			1				
	1		2						
Player 2	2								
	3								
	4								
	5								
	6								
	7								

Now to calculate the probability of winning we can use these two rules

- For a probability set of distinct events you add the individual probabilities
- For the probability of 2 independent events happening in turn you multiply probabilities together

Domino tilings

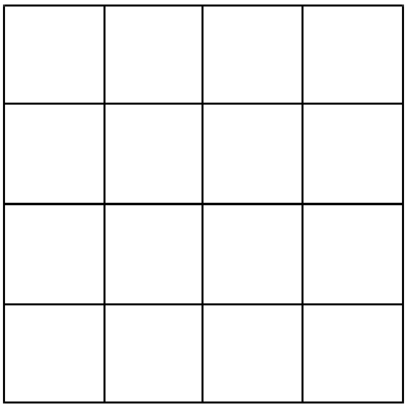
Tile (cover) the following arrangements of squares with 2x1 dominos like these:



Aim for tilings with no overlap between the dominos, and with no square left out.

Can you tile the shapes? If you cannot, why not? Can you find some further shapes that can, and that cannot, be tiled?

a.



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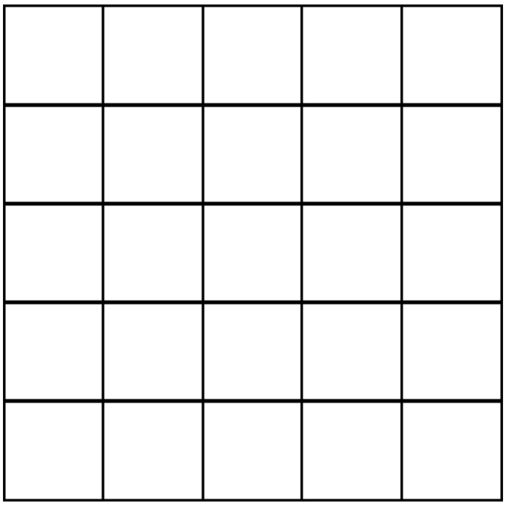

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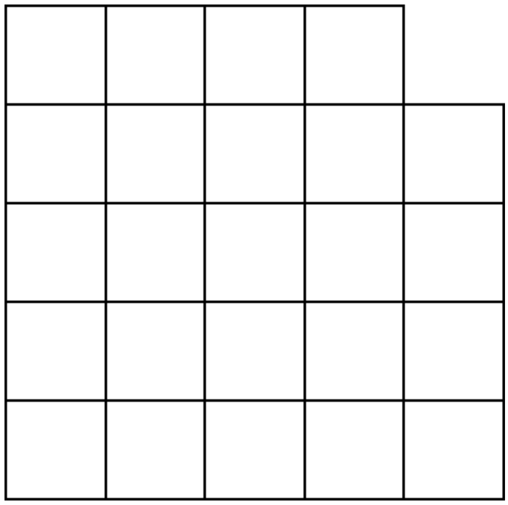
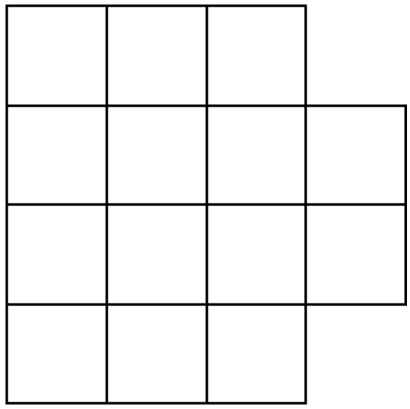
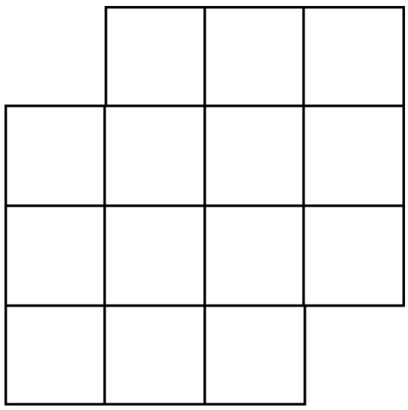


d.



e.



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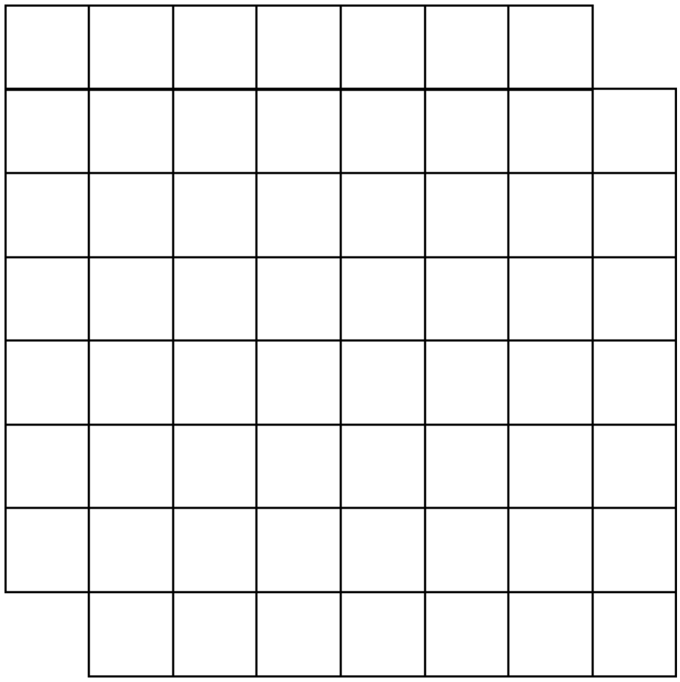


Collatz conjecture

Think of a positive integer

If it is even, halve it

It if is odd, multiply by 3 and add 1

Repeat this process with your new number

Keep going, but stop if you get to the number 1

For example, if you chose the number 6 you would get this sequence:

6 → **3** → **10** → **5** → **16** → **8** → **4** → **2** → **1**

Try some numbers of your own.

Can you find some short sequences?

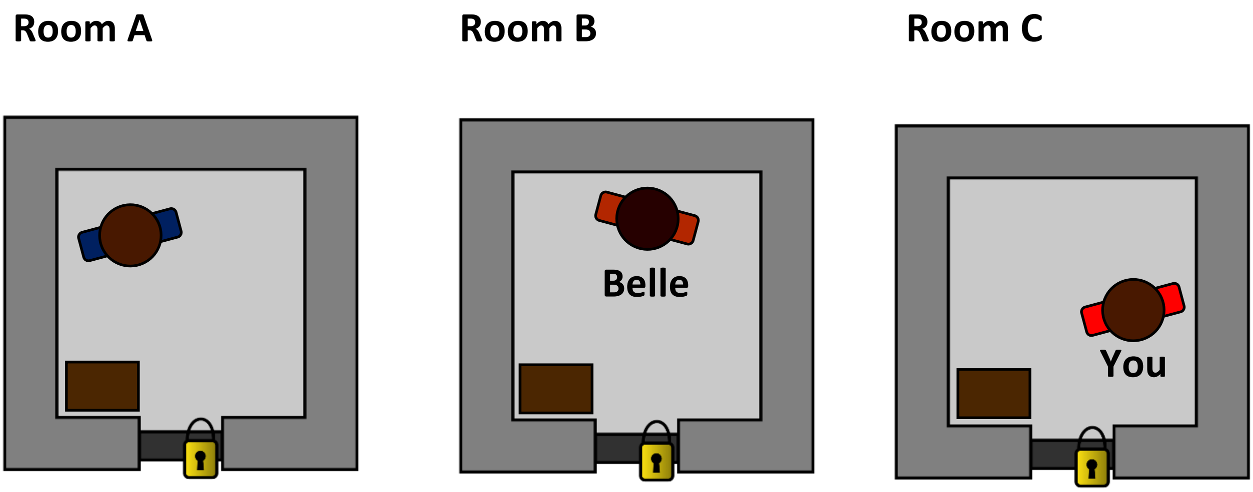
Can you find some long sequences?

Do you always reach the number 1?

Apple Teaser

You and your two friends, Adam and Belle, are captured by an evil gang of apple thieves. In order to be released, the gang's leader, Ruben, sets you this fearsome challenge.

Adam is locked in room A, Belle is locked in room B and you are locked in room C.



You each have a box of apples in your room. You can count the number of apples in your own box, but not in anyone else's.

- Each room has at least one apple
- No room has more than nine apples
- Each room has a different number of apples

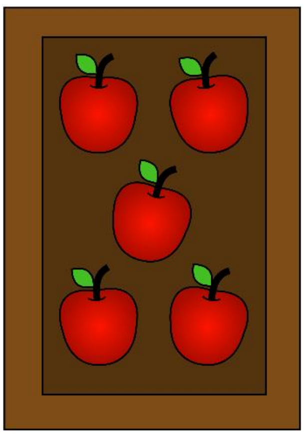
The rules of the challenge are: The three of you will get to ask Ruben one question, which he will answer truthfully with "Yes" or "No". Everyone hears the questions and the answers. Ruben will release you if **one of you** tells him the total number of apples in the three rooms.

Adam: "Is the total number of apples an even number?"

Ruben: "No"

Belle: "Is the total number of apples a prime number?"

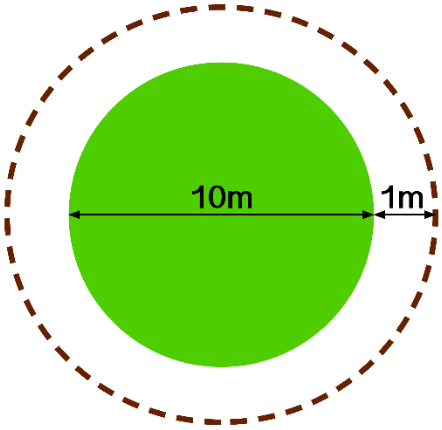
Ruben: "No"



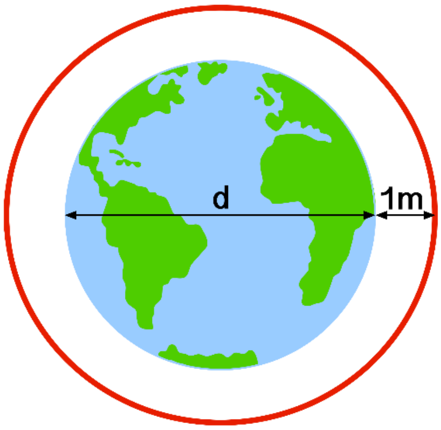
You count five apples in your box. **What question will you ask?**

Circumference

1. A farmer has a circular field which is 10m wide. It has a fence around it to stop his chickens from escaping. If the farmer wanted to extend his field by 1m in all directions, how much more fencing would he need?



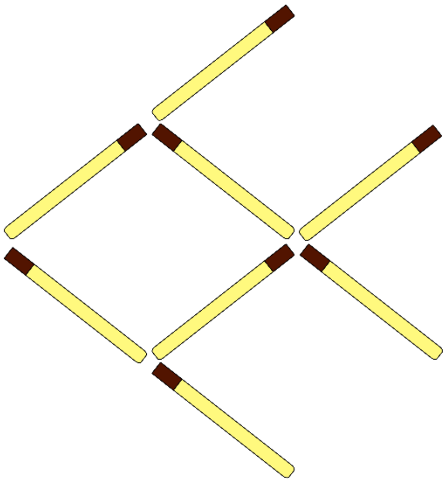
2. Imagine a piece of rope is tied around the equator of the earth. How much longer does the rope have to be if we instead wanted the rope to be 1 metre **above** the earth **at all points**? (The diameter of the earth is d)



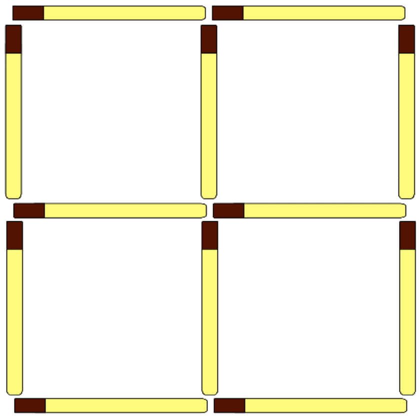
What do you notice about the two lengths?

Matchstick Puzzles

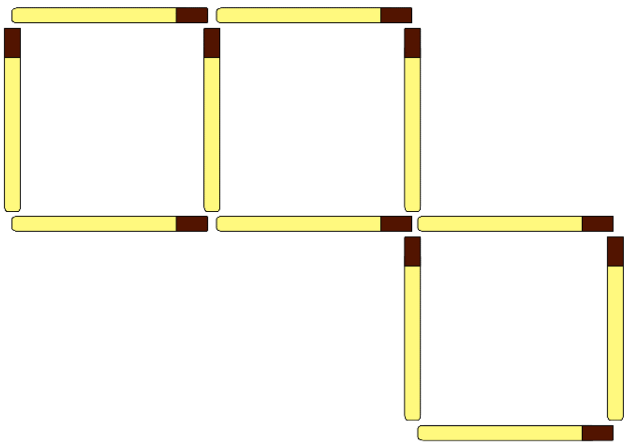
1) Can you reverse the fish by moving only 3 matchsticks?



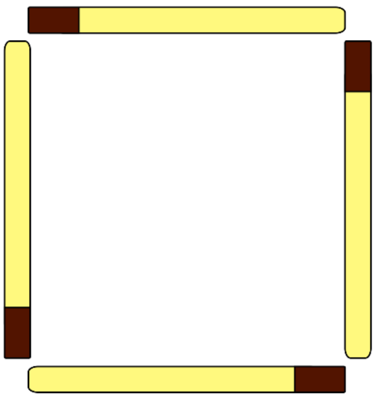
2) Can you make 6 squares by moving only 2 matchsticks?



3) Can you make 2 squares by moving only 3 matchsticks?



4) Can you make 4 triangles and 2 squares by adding 4 matchsticks?



Gabriel's Problem

Gabriel wrote the numbers 1-9 in a 3x3 grid.

He then multiplied together all the numbers in each row and wrote the resulting product next to that row. He also multiplied the numbers in each column together and wrote the product under that column.

He then rubbed out the number 1-9.

			24
			40
			378
60	21	288	

Can you work out where Gabriel originally wrote the numbers 1-9? Did you have more information that you needed?

River Crossing

A farmer bought a wolf, a goat, and a cabbage from the market. To get home he must cross a river by boat, but the boat is very small and he must shuttle the items across one by one.

The wolf cannot be left alone on the same side of the river as the goat (since the goat will be eaten) and the goat cannot be left alone on the same side of the river as the cabbage (since the cabbage will be eaten).

How can the farmer move the wolf, the goat, and the cabbage across the river while making sure that nothing gets eaten while he's away?

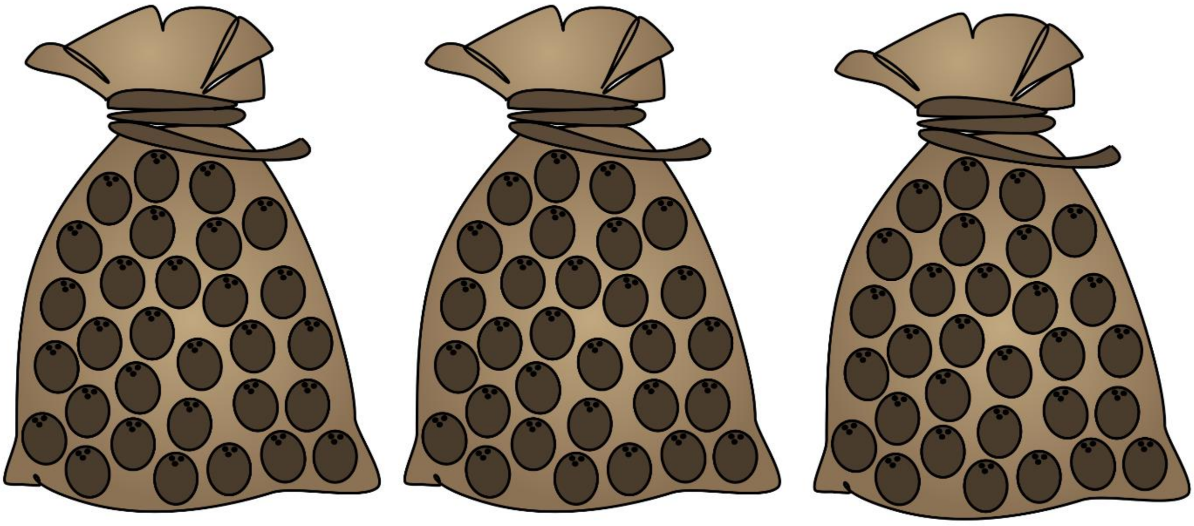
Coconut Trader

You are a coconut trader and you are travelling to the market to sell your coconuts. You are carrying 3 sacks with 30 coconuts in each. No sack can hold more than 30 coconuts.

On the way to the market you have to pass through 30 checkpoints and on each checkpoint you have to give 1 coconut for **each** sack you are carrying.

Once a sack is empty it should be left behind.

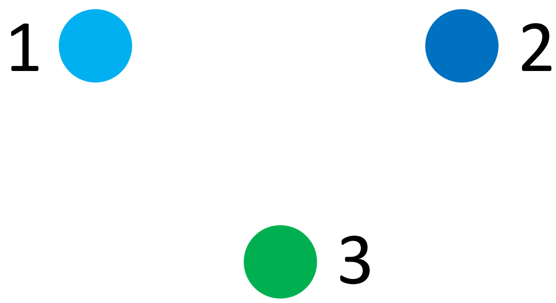
How many coconuts are left in the end?



Handshake Puzzles

Stage 1) If you have **two** people at a party and each person shakes hands with every person once, how many handshakes happen?

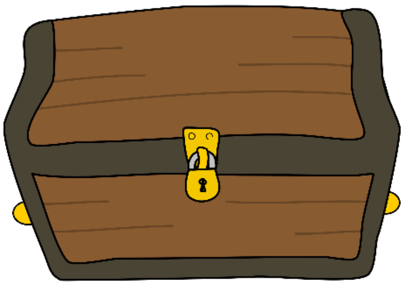
Stage 2) If you have **three** people at a party and each person shakes hands with every person once, how many handshakes happen?



Stage 3) If you have **four** people at a party and each person shakes hands with every person once, how many handshakes happen?

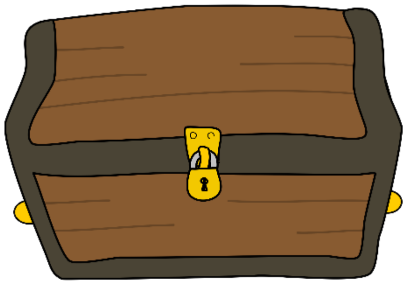
Treasure Hunt

Exactly one of these chests contains treasure, but **only** one of the four statements is true.



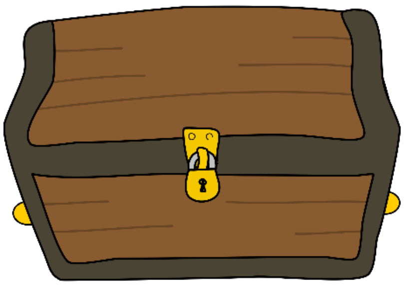
"The treasure is in here"

B



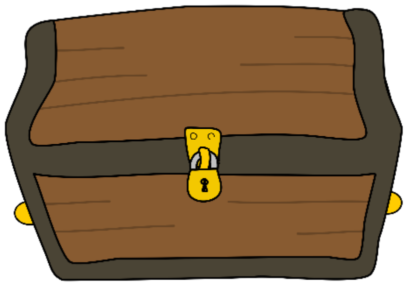
"The treasure is in chest A or D"

C



"The treasure is **not** in here"

D



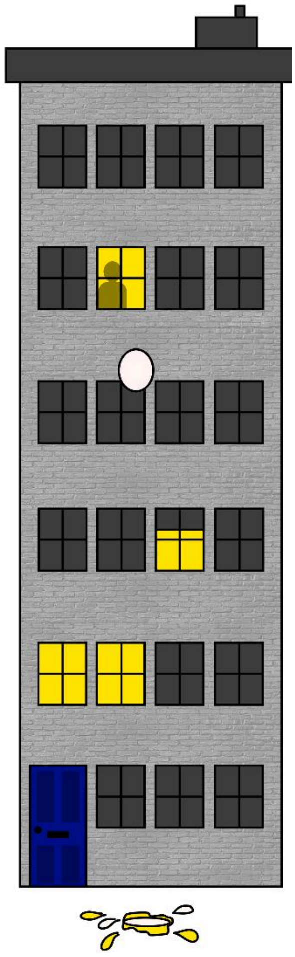
"The treasure is in here"

Which chest must contain the treasure?

Two Eggs

You are given **two eggs** and access to a 100-storey building.

If an egg is dropped and does not break, it can be dropped again. Once the egg breaks, it cannot be used again. If an egg breaks when dropped from floor n, then it would also have broken from any floor above that. If an egg survives a fall, then it will survive any fall shorter than that.

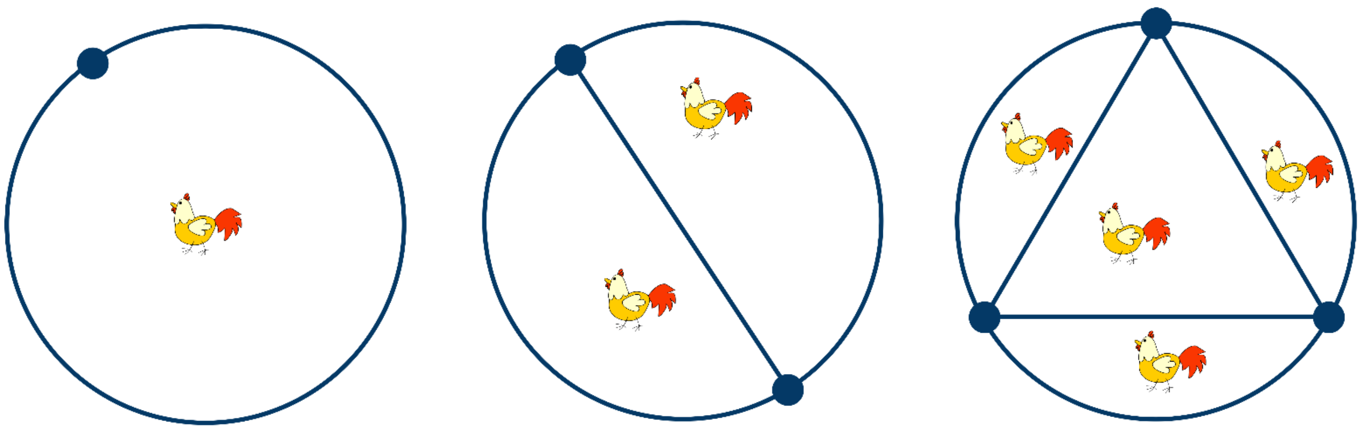


How would you find out what the highest floor is from which an egg will **not** break when dropped out of the window? How would you minimise the number of egg drops it takes to find this out?

Counting Chickens

A farmer keeps his chickens in circular fields, separated by fence posts and straight fencing.

1 fence post → 1 chicken	2 fence posts → 2 chickens	3 fence posts → 4 chickens



Can you work out how many chickens the farmer can keep if he has:

1) 4 fence posts
2) 5 fence posts
3) 6 fence posts

Pell Numbers

It can be proven that there are no integer (whole number) solutions to

$$\sqrt{2} = \frac{a}{b}$$

So there are no integer solutions to

$$2b^2 - a^2 = 0$$

But can you find positive integer values of a and b that *nearly* work? That give the answer 1 or -1?

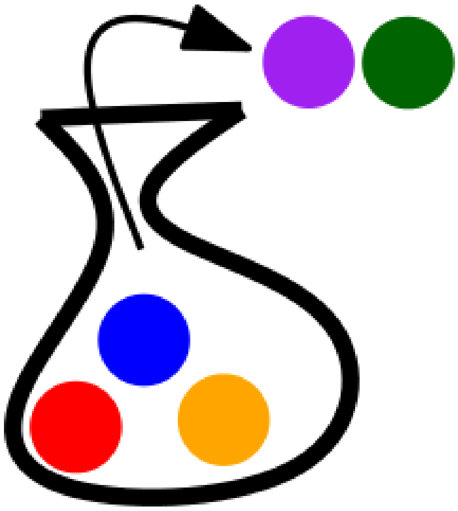
$$2b^2 - a^2 = \pm 1$$

Can you find a pattern to all your solutions?

What does $\frac{a}{b}$ give a good approximation to?

Balls and Books

1. You have a bag of 5 balls, all of different colors. You take two balls out of the bag. How many different possible outcomes are there?

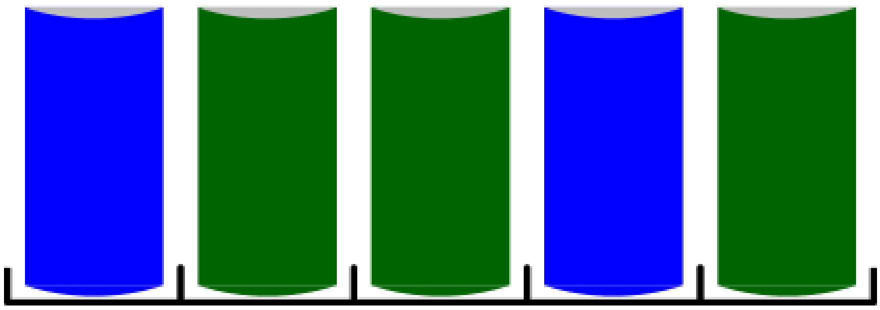


What if you add another ball of a new color to the bag (so you have 6)? What about 7 balls?

Is there a similar problem you've already solved?

2. You have 2 blue books and 3 green books. The blue books all look the same and the green books all look the same.

How many ways are there to line them up in the shelf?



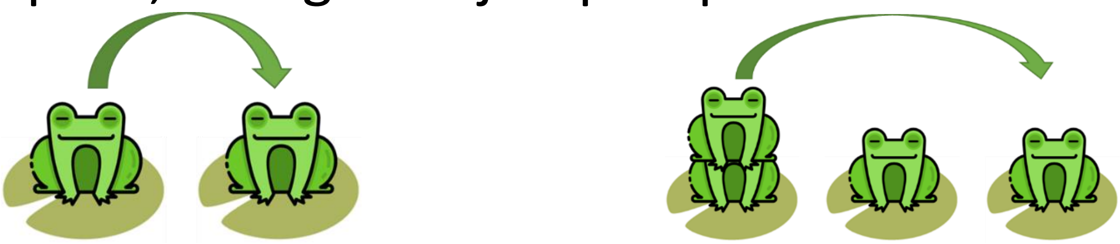
What if there are 4 green books? 5 green books?

Frog Party

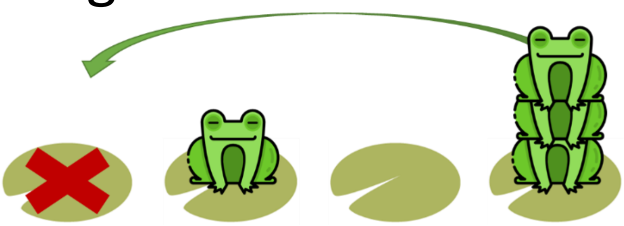
5 frogs want to have a party. To do this they must all be on the same lily pad.

Frogs can jump left and right however there are a couple of rules:

1. Frogs jump together, so that 1 frog can jump 1 space, 2 frogs can jump 2 spaces etc.



2. Frogs cannot land on an empty lily pad



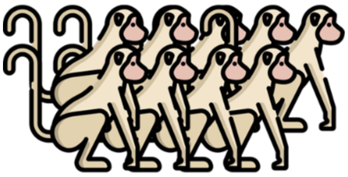
Can you find a way to move the frogs so they can have a party together on the same lily pad?

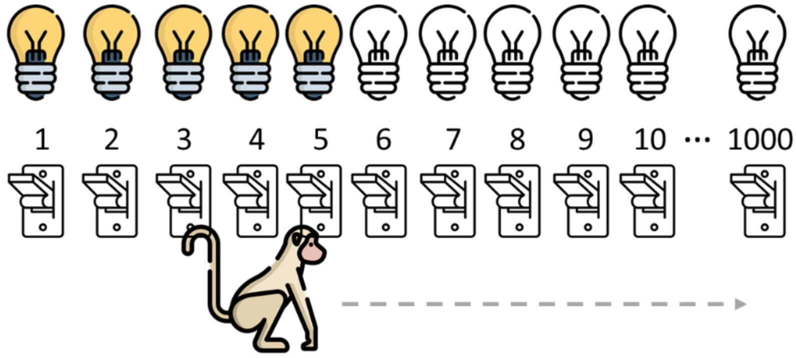
Is it possible to have a party on every lily pad?

Monkey Business

A large room has 1000 lightbulbs in it, all are switched off, but each has its own switch to turn on

1000 monkeys enter the room and decide to press the light switches in a very particular way.





The 1st monkey presses every multiple of 1.
The 2nd monkey presses every multiple of 2.
The 3rd monkey presses every multiple of 3.
Etc., until the 1000th monkey.

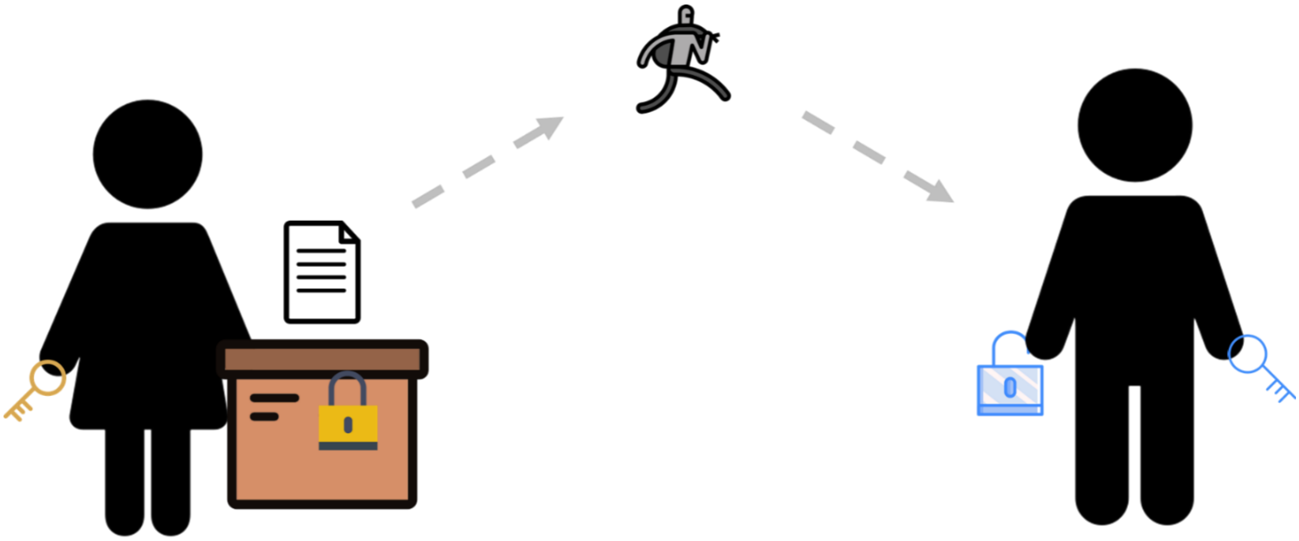
After all the monkeys have finished pressing switches:

1. Will light number 10 be on or off?

2. How many lights in total will be on?

Locks and Keys

Alice has a secret message that she wants to send to Bob, who lives far away. She has a box and a lock and one key. Bob also has a lock with one key.

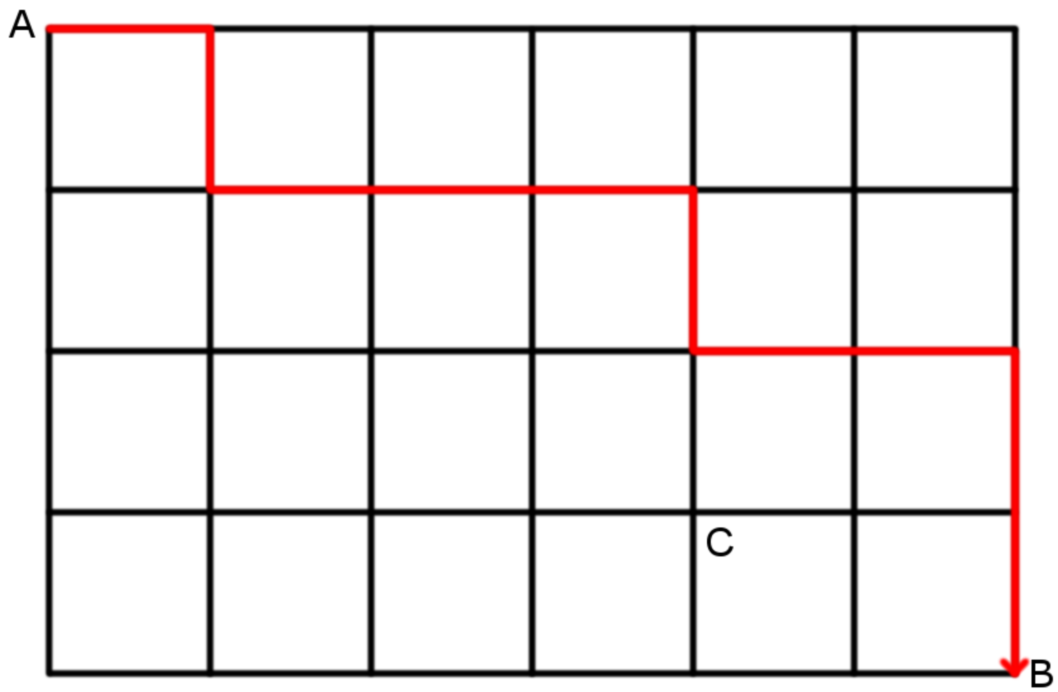


To send things between each other Alice and Bob have a messenger, who can travel multiple times between them. But they don't trust him, and don't want him to read the message.

How can Alice send her message to Bob without the messenger reading it?

Paths

1. How many ways are there from A to B in the grid if you can only walk along the lines and only walk right or down each time? How about from A to C? How about from A to any other point?



2. How can you write down a path from A to B as a sequence of 10 letters? Or as a set of 4 numbers?

Only try these after solving 1 and 2

3. How many ways are there to line up 4 blue and 6 green books in your shelf?

4. If you take 3 balls from a bag of 7 different balls, how many possible different outcomes are there?

Scales and Weights

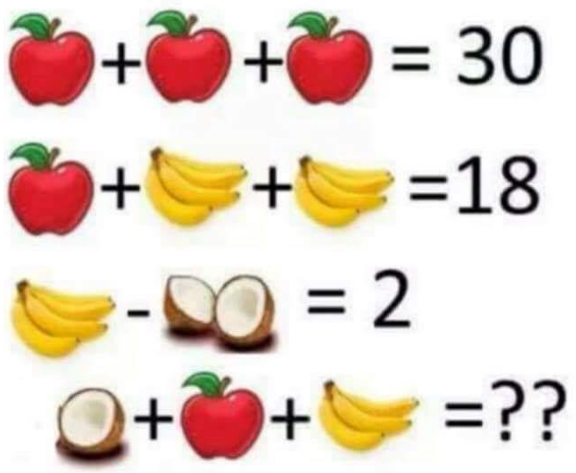
You have a shop selling maize flour by weight. Your customers may want to buy any amount between 1 and 40 kg of flour.

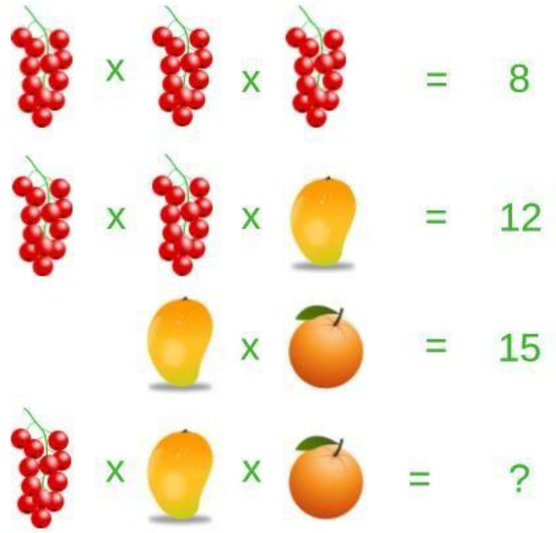
You also have a balancing scale and want to buy weighing stones to use with it.

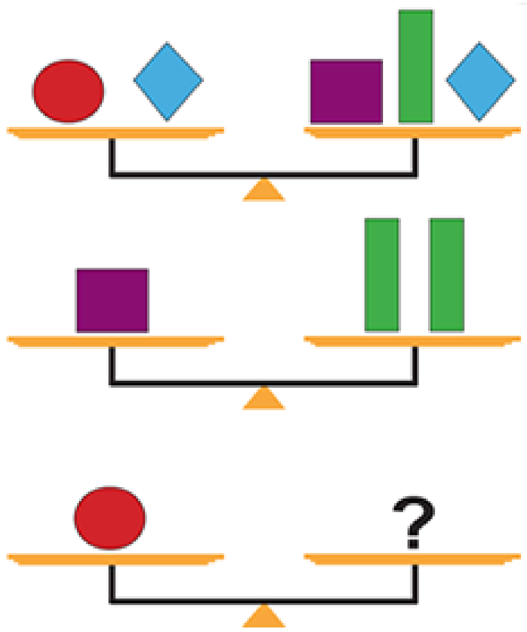
Which weighing stones should you buy so you can measure any weight between 1 and 40 kg? (You want to buy as few as possible)

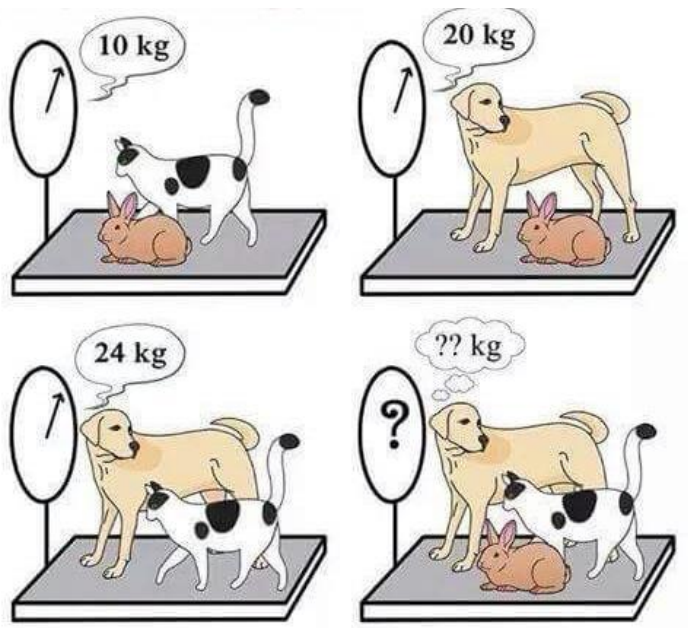
Picture Puzzles I

The following 4 problems all have pictures to represent numbers. Can you figure them out?









Make Many

Task 1

How many different three digit numbers can you make using only the numbers 100, 2 and 3 at most once and as many of the operators $+$ $-$ $\times$ $\div$ and brackets as you like?

e.g. $100 \times (2+3) = 500$

$100 \times 3 = 300$

$100 + 3 + 2 = 105$

Task 2

Using the numbers in the blue box, try and make all of the three digit target numbers. Hopefully Task 1 will have made you think of some tricks for the harder ones!

5 5 6 7 50 100

635

667

665

785

564

420

202

419

Secret Santa

A group of friends want to buy one present each and everyone wants to receive one present (Secret Santa). The rules of a Secret Santa are that each person's name is put in a hat and the names are mixed. Then each person must choose 1 name from the hat. If you choose your own name, you must put it back in the hat (and all start again if necessary!).

If **2 people** do a secret santa there is only one way: Person A gives to person B and person B gives to person A.

3 people: With 3 people, there are 2 possible ways. Can you think why?

4 people: Now how many different ways are there with 4 people?

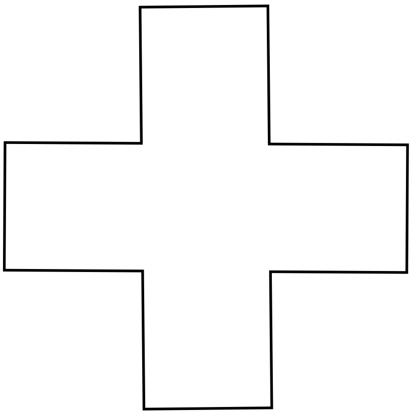
The final challenge is to find the number of different scenarios with **5 people**.

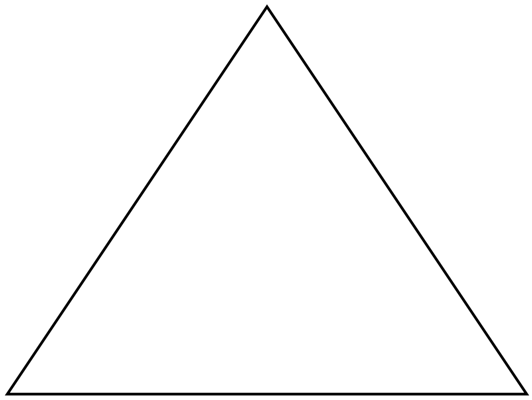
Computer Activities

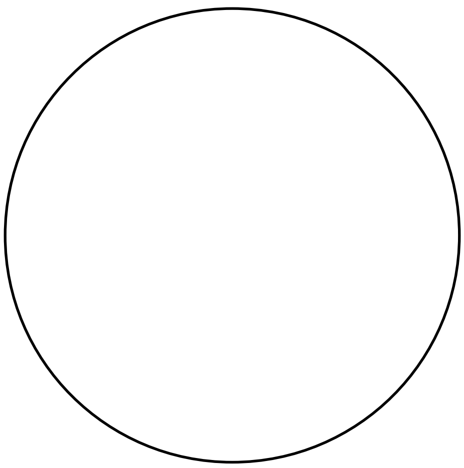
Logo Challenge 1

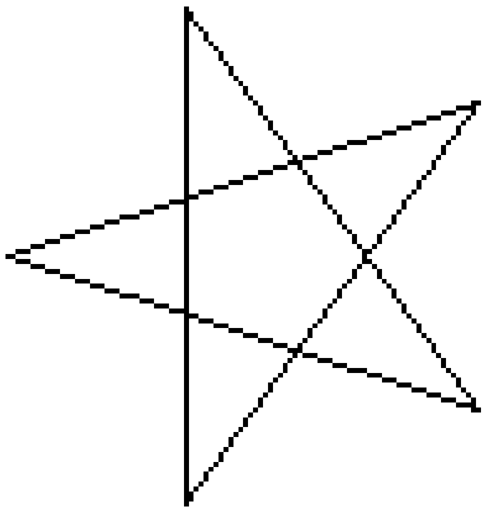
Use MSW Logo on the computer

Draw the following shapes in logo. See who can draw them using the fewest words





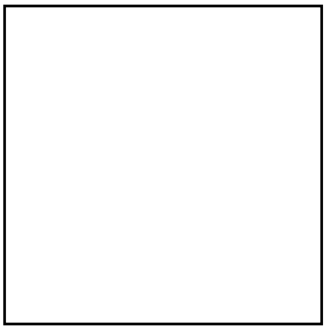




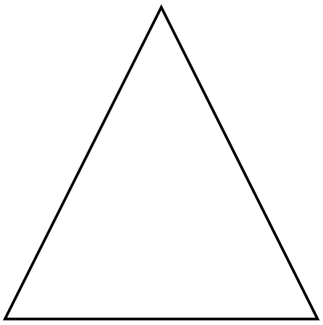
GeoGebra Challenge 1

Use GeoGebra on the computer

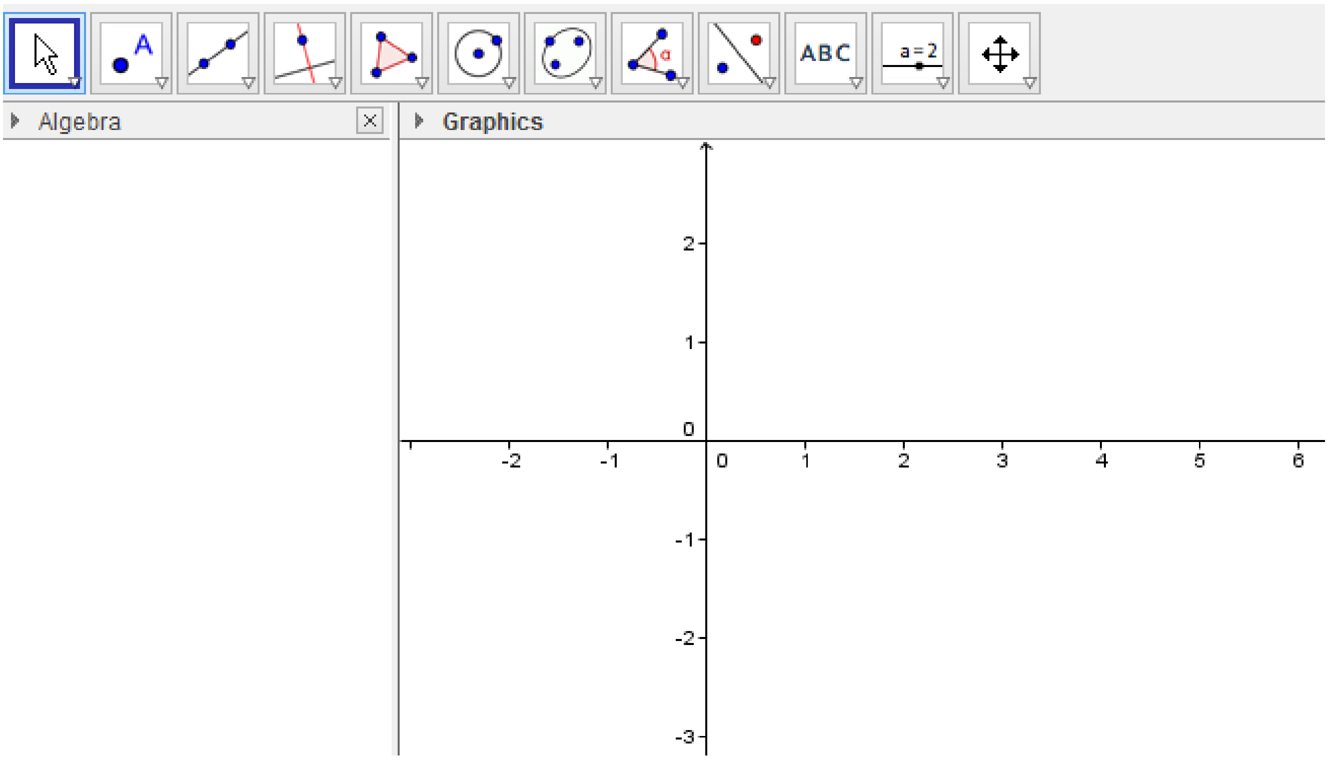
Draw the following shapes. Can you prove that they are correct?



Square

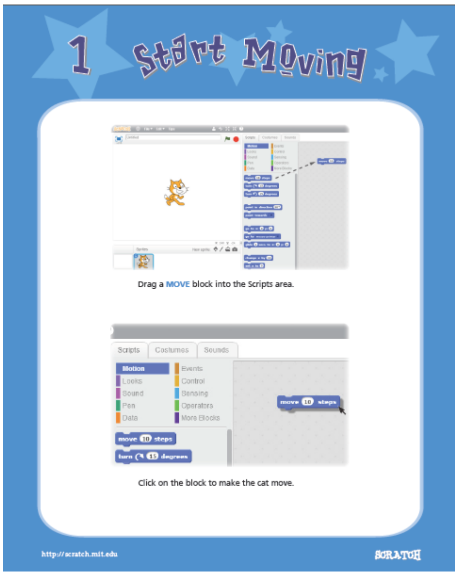


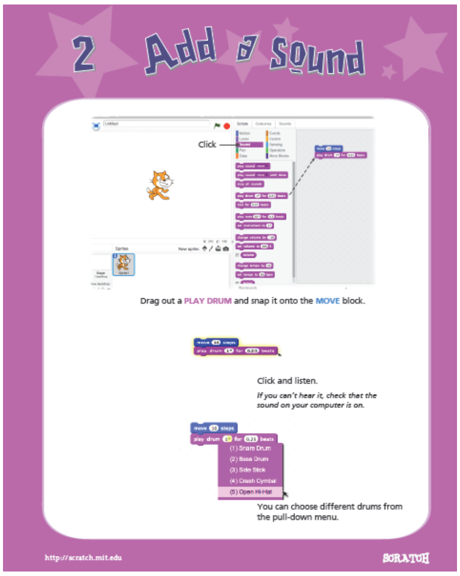
Isosceles Triangle

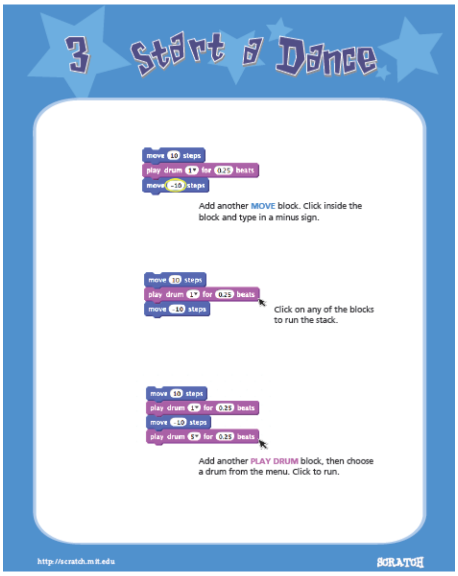


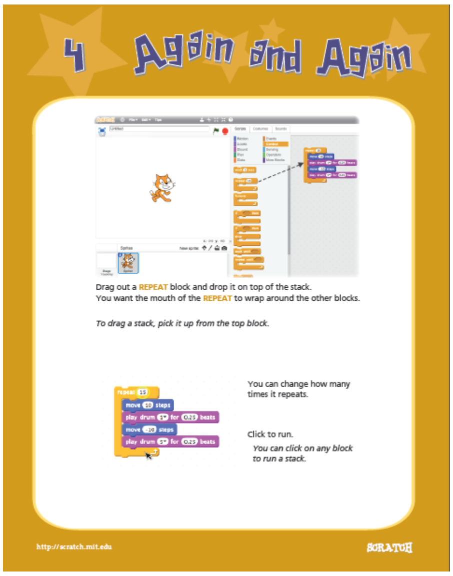
Scratch Challenge 1

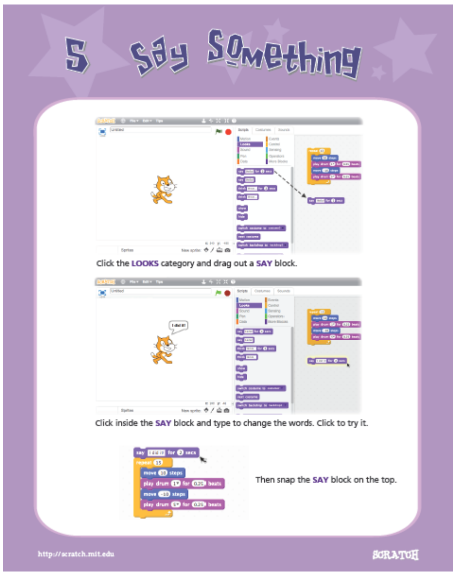
Use Scratch on the computer. Try to work through the tutorial activities which can be seen in full in the facilitator notes section











GeoGebra Challenge 2

1. Draw a line segment, AB in GeoGebra.

2. Construct the **perpendicular bisector** of the segment.

3. Select a point C, anywhere on the perpendicular bisector and use the **intersect** tool to select the intersection point, D of the lines.

4. What does the perpendicular bisector do to the line segment? Use the **Length** and **Angle** tools to check your ideas. e.g. what is the angle ADC or the lengths AD and DB? Move the points A and B and see what happens.

5. Construct the segments AC and BC. How many triangles are there and what types?

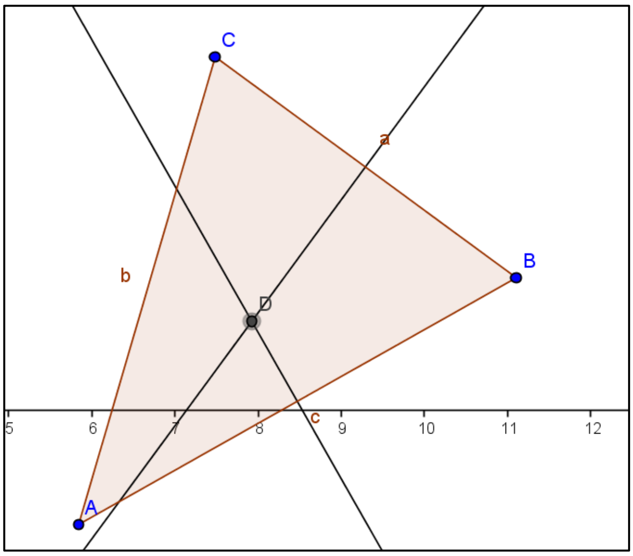
6. What can you say about the lengths AC and BC? Why? Prove this using congruent triangle theorems.

What happens when you move the points around?

GeoGebra Challenge 3

1. Draw any triangle ABC in GeoGebra.

2. Construct the **perpendicular bisector** of the segments AB and BC.



3. Use the **intersect** tool to select the intersection point, D of the two lines.

4. What can you say about the lengths of AD and BD? Why? Use the **length** tool to check your answer.

Is the same true for BD and CD? Why?

5. Using your answers to 4., what can you now say about AD and CD?

6. Using your answer to 5., what will you see if you construct the perpendicular bisector of AC? Try it.

7. Is the point D always inside the triangle? If not, when is it? Use the **move** tool to move points A, B and C.

8. What will happen if you construct a circle with centre D through A? If you understand 5., you will be able to answer this. D is called the **cicumcentre** of the triangle.

Facilitator Versions

Games

Tic Tac Toe/O-X

This game is designed to get students to think about strategy when play games. After students have played the game for a while ask them some of these questions.

- What is the best way to play this game?
- Is it better to go first or second?
- Where is the best position to start from?
- Are the rules the only thing you need to know to win the game or is there something more?
- What is strategy and how does it relate to this game?
- What are some of the strategies to playing this game?
- Should you play this game randomly or should you think about your moves?

After discussing some of these things get the students to play again, and this time tell them to think about strategy and think about your moves.

After playing again they should find they are getting more draws. Ask them why this is.

If you are getting lots of draws it is because both players have become better at the game and have learnt how to play better.

To be good at this game requires mathematical thinking. You have to learn the strategies to become good at the game through practice and by thinking carefully about your moves. When you lose a game you should try to understand why and make sure it doesn't happen next time.

Next thing to do

After doing this activity or in your next session you can try the activity 15 game. It is another game that requires this type of thinking. Follow the instructions on the sheet.

15 Game

Introduction

You should only play this game after you have played Tic-tac-toe!

This games requires that you can do some mental calculations and you can think ahead and predict what your opponent will do. Like Tic-tac-toe you need to develop a strategy if you want to win this game.

It is usually harder for students to develop a strategy for this game than for Tic-tac-toe. Ask the students whether they think this game is easier or harder than Tic-tac-toe, and why.

Solution

Discuss with the students the differences between playing the game in a line and playing in the square. Discuss these questions.

- Is it easier to play the game in the square or in a line? Why?
- Do you get more wins, losses or draws?
- What is special about the arrangement of the square?
- Is playing in the square similar to anything you have played before?

After playing in the square a few times, students should realise that to get 15 they just need to get a line of three numbers in a row. This is because if you look at the numbers in each horizontal, vertical and diagonal lines, they all add up to 15. That is why this is a special arrangement of the numbers, it is called a **magic square**.

We have now transformed the game into getting a line of 3 numbers. This means that it is basically the same game as **Tic-tac-toe**!

Students usually say that Tic-tac-toe is an easier game to play than 15 Game, but we have shown that we can transform this game so that we are basically playing Tic-tac-toe again.

In mathematics we often change the way we look at things to make them easier for us. It is a very powerful tool. Here we have transformed a hard game (15 Game) into an easy one (Tic-tac-toe). With this knowledge, students should be much better than the 15 Game now.

Extension

Are there any other arrangements of the numbers that will make a magic square? How many are there?

Patience

After one team wins the first round, ask the winning team to explain how they won.

Ask the others teams what they could do differently to turn over the cards faster. Get them to think about strategy.

Play the game a few times until the students get the idea of strategically replacing incorrect cards in order to minimize the number of times the same cards are turned over. With 10 cards, the best strategy will accomplish the task in 19 moves or less, with a minimum of 10 turns if there are no mistakes. With no strategy, the game can go on forever.

Students should understand that just going very fast is not the best method. If they have a good strategy, they can finish faster than a team running fast without a strategy.

There may be many different strategies used by different teams. It is good to have a discussion about different strategies and think which ones will be better than others.

Mastermind

Introduction

This game tests logic and reasoning skills. Students will first guess the code at random, but should be able to develop strategies to help them guess in a faster time. An example of how the game might work to crack the code RRGG using a good strategy is outlined below:

Code	Code-breaker Guess	Correct colour Correct position	Correct colour Wrong position
RRGG	RGBY	1	1

The player now knows that there are only 2 different colours used in the code because otherwise they would have more *'correct colour wrong* position'. Using this they might guess:

Code	Code-breaker Guess	Correct colour Correct position	Correct colour Wrong position
RRGG	RGBG	2	1

The only letter that changed was Y->G in position 4. There is another colour in *'correct colour correct* position', therefore the G must belong in position 4. The next guess is:

Code	Code-breaker Guess	Correct colour Correct position	Correct colour Wrong position
RRGG	GRBG	1	2

The next guess is:

Code	Code-breaker Guess	Correct colour Correct position	Correct colour Wrong position
RRGG	RGGG	3	0

The fifth guess is:

Code	Code-breaker Guess	Correct colour Correct position	Correct colour Wrong position
RRGG	RRGG	4	0

Solution

One good strategy for beginners is to change only one colour or one position each time a guess is made. This is not however the optimal strategy.

Extension

Try increasing the number of colours and positions, be careful though as the game quickly becomes very difficult.

Nim/21

Introduction

At first students will play the game randomly without thinking. After everyone has played a few times tell them to play again but really think about what numbers they say and try their best not to lose. The game can seem like luck but if you know how to play you can never lose.

This activity is all about thinking and strategy. If you have a good strategy you have a better chance of winning. Do not just play this game a few times, you need to play a lot to work out the best way to play.

Solution

It is obvious that if you stop at 20, you will win. After playing a few times students realise that if you stop at 16 you cannot lose. This is because, after you say 16, the next player can say 17, 17-18 or 17-18-19. But wherever they stop you can always get to 20 on your next turn and win. If you stop at 16 your opponent can't get to 20.

How can we make sure we get to 16 before the other player?

If students realise they need to get to 16, tell them to play again and ask them to think about how they can make sure they get to 16 before their opponent. Think about what we did to get from 16 to 20 and try to do the same. You should realise that if you get to 12 and stop, you will be the player who says 16, and win the game. (If you stop at 12 the other player can only get to 15.)

So you need to get to 12 before you opponent to win. 12-16-20. Do you see the pattern now?

We can see that we want to be the player who says 12, 16 and 20, then we will win. If we keep going like this we see that to make sure this happens we just need to say all the multiples of 4. If we say 4-8-12-16-20 we will win every time. So to win the game we just need to get to 4 and stop. Then we can get to 8 on the next turn and stop, and so on up to 20.

How can we make sure we get to 4 before the other player?

If you want to get to 4, should you go first or second? If you go second your opponent can only go up to 3 and you can stop at 4 on your turn. So to be sure to win the game, you want to go second.

Once you know this strategy you will never lose this game, unless you opponent also knows and decides to go second! At first this game seems like luck, but by thinking you can create a clever strategy to win. This is what we call ***Mathematical Thinking***.

Extension

There are many different versions of Nim/21 on a separate sheet. One variation is to change the rules so that the player who says 21 is now the **winner**. How does this change the strategy you use?

Nim-Related Games

Introduction

Make sure students have first played the Nim/21 game and are familiar with the strategies for winning and losing.

The variants on the game Nim have many different strategies, some simple and some complex. See if any students can consistently beat others and ask them what their strategy is.

Solution

There is no simple solution to understand the optimal strategy for Nim, however by thinking strategically and in a way similar to the 21 game players will have an advantage over their opponents. A working solution using binary is presented below, however it is not expected that students will discover it or fully understand why it works. It might make for a nice extension activity to see if it actually works.

Assume that the established rule is that the person who picks up the last object wins. You are the first player. Write the binary representation for the number of objects in each pile. For example, if there are 3, 4 and 5 objects in piles 1, 2 and 3, respectively, write these as

Pile 1 011

Pile 2 100

Pile 3 101

Then, add the numbers as if they are decimals, to get 212. A winning position is one that has all even digits. To get to a winning position, change the 1 to a zero by taking two objects (binary 10) from Pile 1, leaving one object (binary 001). The second player will inevitably have to go to a losing position, which you can then turn into a winning position.

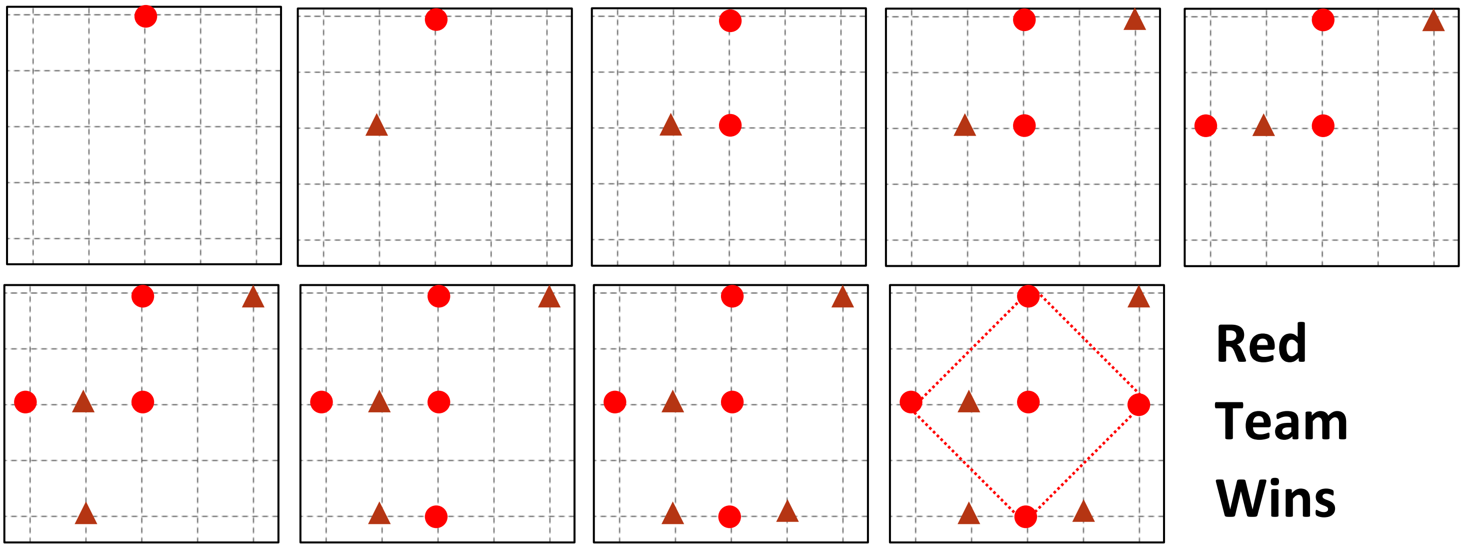
Extension

Try to find winning strategies for other variants of the Nim game, or test the binary solution above to see if it works for standard Nim.

Making Squares

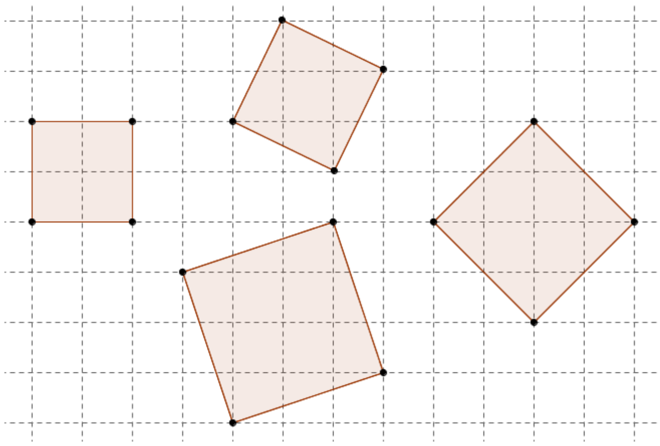
Introduction

This game is a great way to practice using coordinates and develop geometrical reasoning (understanding of shapes). It might be a good idea to play the game once altogether, splitting the class into two teams. Each team takes it in turns to mark a point on the game grid until one team can join four of their own points together and make a square. A typical game on a smaller grid might look like:



Solution

There are lots of strategies that teams can use, winning teams are most likely to be those who can visualize squares which have been rotated also.



Part of the challenge of the game will be to convince all the other players that you have identified a correct square

Learners may need to think about the properties of squares to prove it when they think they have won.

Extension

If using coordinates you could tell students they must call out the coordinate correctly before marking the point. You could also try using other shapes such as parallelograms, or holding a small contest with students playing in pairs.

Paper Pieces

Introduction

This game is designed to get students to think about winning strategy when playing games, and introduce the idea of strategy stealing.

After students have played the game for a while ask them some of these questions.

- Did the first or the second player tend to win more often?
- Is it better to go first or second?
- Is there an easy strategy for one player to always win?

After discussing some of these things get the students to play again, and this time ask them to see if they can come up with a strategy that always wins for one of the players.

Solution

There is a winning strategy for the second player who can just copy the first player's moves, i.e. if the first player fills two cells in the first/second row, the second player just fills out the cells below/above in the second/first row respectively. It's always possible for the second player to do such a move, and thus it will always be the first player who at some point can't make a move anymore. Having a winning strategy that just copies the other player's moves is called strategy stealing.

Variation

Ask the students to play the game on a strip that has an even number of cells first.

After having played for a while, ask the same three questions as for the first version of the game. If they've seen the winning strategy for the first version of the game, ask if they can somehow use a similar strategy for the variation.

The winning strategy here is for the first player to mark the two cells in the middle of the strip. Then we are left with two identical pieces of the strip to the left and to the right of these marked cells. Now when the second player makes a move on the right/left half, the first player can just copy that move on the left/right respectively in the same vein as in the winning strategy for the original game.

Extension

You can have the students also play the game on the paper-strip with an odd number of cells. However, in this case there is no easy winning strategy. You can ask them to play the game on strips with small numbers of cells, and see for which numbers the first player and for which numbers the second player can always win. (Example: for 3 and 7 cells the first player always wins, for 5 cells the second player always wins. For 9 cells the second player can always win, but the analysis is a bit more complicated because the first player can win if the second player doesn't react correctly to the first player's move.)

Tic Tac Toe with levels

Introduction

This game is best played after playing the simpler Tic-Tac-Toe (G1 sheet). This is a variation of Tic-Tac-Toe that has much more variation. This game is designed to get students to think about having a strategy to win the game. After students have played the game for a while ask them some of these questions.

- What is the best way to play this game?
- Is it better to go first or second?
- Where is the best position to start from?
- Are the rules the only thing you need to know to win the game or is there something more?
- What is strategy and how does it relate to this game?
- What are some of the strategies to playing this game?
- Should you play this game randomly or should you think about your moves?

If you have previously played original Tic-Tac-Toe, ask the students what are the similarities and differences between this game and Tic-Tac-Toe. Some questions to discuss include:

- Which game is easier to win? Why?
- Which game requires more thinking, and why?
- Are the strategies you thought of for both games similar in any way? How?

Student should realise that this game is much more complex than original Tic-Tac-Toe which has a clear strategy to avoid losing. In fact, even a computer analysing the game could not find a winning strategy for Tic-Tac-Toe with levels. This should show students how even simple games can require lots of mathematical thinking, strategy and logic.

Extension

Add another row so you now have a 4 x 3 grid. Use the same rules. Does this make the game harder or easier?

Next thing to do

After doing this activity or in your next session you can try the activity 15 game, original Tic-Tac-Toe or Letter matching, which are all games related to this game.

Letter Matching

Introduction

You should only play this game after you have played Tic-tac-toe!

This games requires that you to keep track of yours and your opponents words, think ahead and predict what your opponent will do. Like Tic-tac-toe you need to develop a strategy if you want to win this game.

It is usually harder for students to develop a strategy for this game than for Tic-tac-toe. Ask the students whether they think this game is easier or harder than Tic-tac-toe, and why.

Solution

Discuss with the students the differences between playing the game in a line and playing in the square. Discuss these questions.

- Is it easier to play the game in the square or in a line? Why?
- Do you get more wins, losses or draws?
- What is special about the arrangement of the square?
- Is playing in the square similar to anything you have played before?

If you have also played 15 game you can see if students notice the similarities with this game (they are basically the same game, called isomorphic games). After playing in the square a few times, students should realise that to get three words that share a letter they just need to get three words in a line. This is because if you look at the words in each horizontal, vertical and diagonal line, they all contain a common letter.

	eat	bee	less	→e
	air	bits	lip	→i
	soda	book	lot	→o
↙	↓	↓	↓	↘
s	a	b	l	t

That is why this is a special arrangement of the letters!

We have now transformed the game into getting a line of 3 words. This means that it is basically the same game (isomorphic) as **Tic-tac-toe**!

Students usually say that Tic-tac-toe is an easier game to play than this game, but we have shown that we can transform this game so that we are basically playing Tic-tac-toe again and made the game easy.

In mathematics we often change the way we look at things to make them easier for us. It is a very powerful tool. Here we have transformed a hard game (Letter matching) into an easy one (Tic-tac-toe). With this knowledge, students should be much better at Letter Matching now.

Extension

Can students come up with their own words to play this game? They must be able to write them in a square like above for the game to work!

Tangrams

Introduction

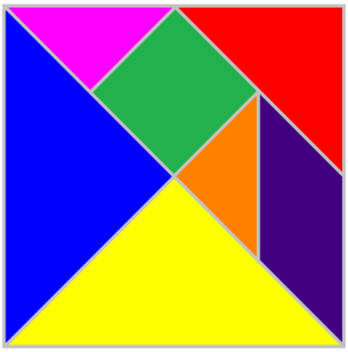
Tangrams were invented more than 200 years ago, and consist of 7 shapes joined to make a square:

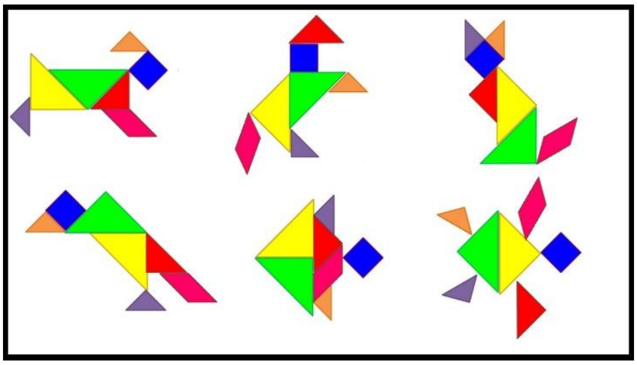
- ☐ 2 large right triangles
- ☐ 1 medium-sized right triangle
- ☐ 2 small right triangles
- ☐ 1 small square
- ☐ 1 parallelogram

They are a great way to explore these shapes, and to also develop the mathematical skill of 'spatial reasoning' (thinking about objects and the space they take up and understanding how they could be moved and manipulated).

Solution

There is no real winning or solution, however, here are some ideas of the types of things people might be able to create





Extension

You could challenge students to recreate a picture by following instructions. One student creates a picture and hides it. They then have to tell another student steps to recreate it, and see if they match at the end (the instructor can't touch the tangram of the creator.

You could make even more shapes by using 2 or more sets of tangrams together.

Countdown

Introduction

This is a really fun and fast game. To prepare you will want to have the cards from 1-10 cut out (2 copies) and 25, 50, 75, 100 (1 copy). Keep the small number cards in a separate pile to the larger number cards.

For the first few games you might want to intentionally pick cards with easier numbers, and a smaller target. E.g. numbers 2,3,5,8,10,100 and target 253

Students do not need to use all of the cards to reach the target, but they can only use each card at most once!

There should somebody keeping a timer. The idea is not to spend a long time trying to reach the target exactly, but to see who can get closest in a short amount of time. If 30 seconds is too short you could consider 1 minute.

At the end of each round the winner is decided by whoever has a number closest to the target. They should be able to prove their solution, by writing on the board. If they have made a mistake you should go to the next closest person.

Solution

There is no solution, and not every game will be possible but there are some general strategies to encourage

1. Use the large number to get as close as possible first
   E.g. with numbers 2,3,4,6,8,50 and target **<u>423</u>**, think of how you might get to 400 using 50.
   If we start with 50 x 8 = 400 we are quite close

2. You can manipulate the large number before multiplication to make small changes. In the above example we know that 50 x 8 = 400, but we could also add a number to the 50 to increase the total.
   I.e. 50 x 8 = 400

   but (50+2) x 8

   = (50 x 8) + (2 x 8)

   = 400 + 32 = 416

   Which is getting close to our target of 423! and we still have 3, 4 and 6 to use.

Extension

After a few rounds of picking cards intentionally you could let the winner pick cards for the next game. The facilitator could still pick the target, or all the cards could be put face down to make it more random.

You could also choose only 4 small numbers alongside 2 large numbers for increased challenge

Problems & Puzzles

Monty Hall Problem

Introduction

Two students should do a demonstration first.

Use playing cards or cut cardboard into three equal cards about 8 cm x 12 cm. On one side of two of the cards, draw a goat (or use a low valued playing card). On one side of the other card, draw a car (or use a high valued playing card). Make sure that the cards look exactly the same on the reverse side. The game is won by choosing the car (the high valued playing card).

Place the cards face down. One student, the player, chooses a card but does not turn it over. Another student looks at the two remaining cards and turns over one with a goat (a low valued card). A card with a goat (low value) is now face up and the other two cards are face down. The player is then given the opportunity to switch his choice to the face down card he had not chosen before or stick with his original card. The class members should be asked whether they would switch or stick with the original card or if it doesn't matter (i.e. if they think the player's chances of winning are the same whether he sticks or switches). The card the player chooses is turned over and if it is a car (the high valued card), he wins.

The class should then divide into pairs and repeat this experiment many times. Each pair of students should draw the following chart to record their results.

Strategy/Outcomes	Switch	Stick
Wins		
Losses		

Each pair should record on the chart the number of times each of them switched and won, switched and lost, stuck and won, and stuck and lost. Then the number of times the class members switched and won should be divided by the total number of times they switched to find the probability of winning if the player switched cards. A similar calculation should be done for the probability of winning if a player stuck with the original choice. Calculate the totals for the class. Ask the class now whether they think it is better to switch or stick? Why is this so?

Notes

This is a problem about probability so you might want to start by asking the class a few questions:

- What is probability? (The number of times an event can happen divided by the total number of possible outcomes)
- How is it measured? (In fractions or percentages)
- If you express probability as a fraction, what is the sum of all of the possible outcomes? (One)
- If you toss a coin, what is the probability of getting a head? (1/2 or 50%)
- If you roll a die, what is the probability of rolling a six? (1/6 or 16.6%)
- If you toss a coin two times, are you guaranteed to get one head and one tail? (No)
- Why not? (There is a difference between theoretical probability, in this case ½, and experimental probability. If you keep tossing a coin and recording the results, the experimental probability will get closer and closer to the theoretical probability)

After the class records its results, you should find the probability of winning is about 2/3 if you switch cards and 1/3 if you stick with your original choice. There are several ways of explaining this:

1. In the beginning, the player had a 1/3 chance of picking the car. If he sticks with that choice, the probability of the card having a car never changes. As probabilities must add up to one, the other two cards together have a 2/3 chance of having the car. After one of the cards depicting a goat is turned over, the remaining card now has the whole 2/3 chance to have the car, as the card that was turned over has no chance of being a car. The probabilities still have to add up to 1. So by turning over a card with a goat, the chances have now increased that the remaining card will have the car. If the player switches to that card, his chance of getting the car is now 2/3.

2. Let's look at all of the possibilities. The car could be on the first, second or third card. If the car is on the first card and you pick the first card and stick with it, you win. If you pick the first card but then switch, you lose. If the car is on the second card and you pick the first card and stick, you lose. If you switch, you win. Lastly, if the car is on the third card, if you pick the first card and stick with it, you lose. If you switch, you win. Totaling all of the possibilities, if you switched, you win 2/3 of the time and if you stuck with your original choice, you only win 1/3 of the time.

Explain to the class that the more information you are given, the better your chances are of choosing correctly. If you saw two cars racing and you knew one was a very fast car but didn't know anything about the other one, your odds of choosing the winning car are better than if you knew nothing about either car. In the Monty Hall game, the player has been given more information once the other person has looked at the two remaining cards and turned over a goat card.

Solution

#1 The Monty Hall Problem

The best strategy with the game is to **switch.** Every time.

- A player whose strategy is to switch every single time **will only lose when the door they initially selected had the car behind it.**
- Since the odds of choosing the car on the first move are one in three, the odds of losing the game when you switch every time are also one in three.
- This means that a person who switches every time will win two-thirds of the time.
- This is double the odds of winning of the person whose strategy is to stay every time.

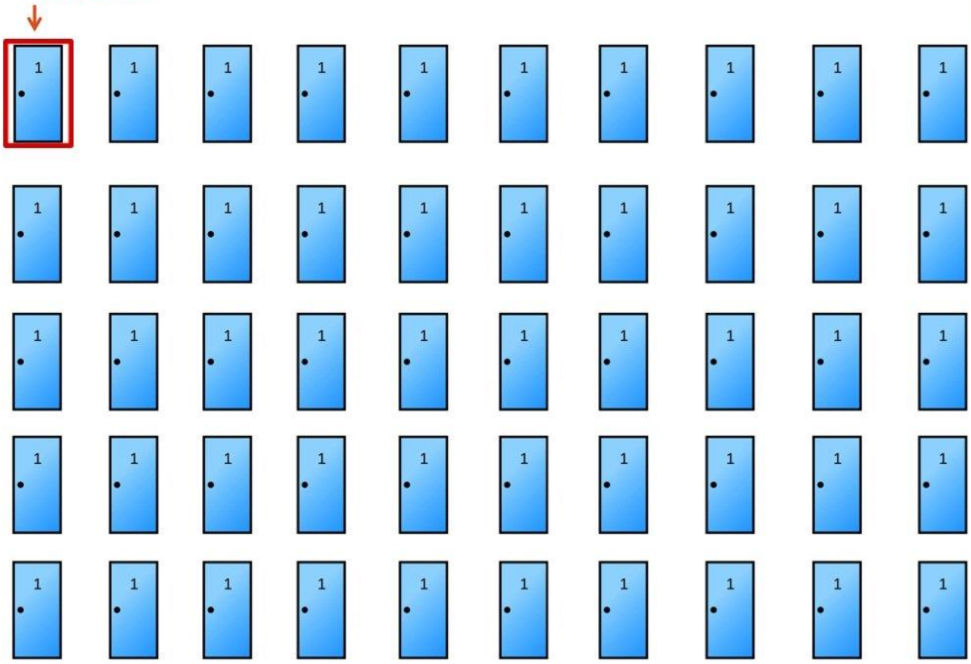
Still don't believe me? Let's say you picked door #1. Here are all the possibilities of what could happen:

Door # 1	Door # 2	Door # 3	Result if stay with # 1	Result if switch
Car	Goat	Goat	**Car**	*Goat*
Goat	Car	Goat	*Goat*	**Car**
Goat	Goat	Car	*Goat*	**Car**

If you stay, you win one in three times. If you switch, you win two in three times.

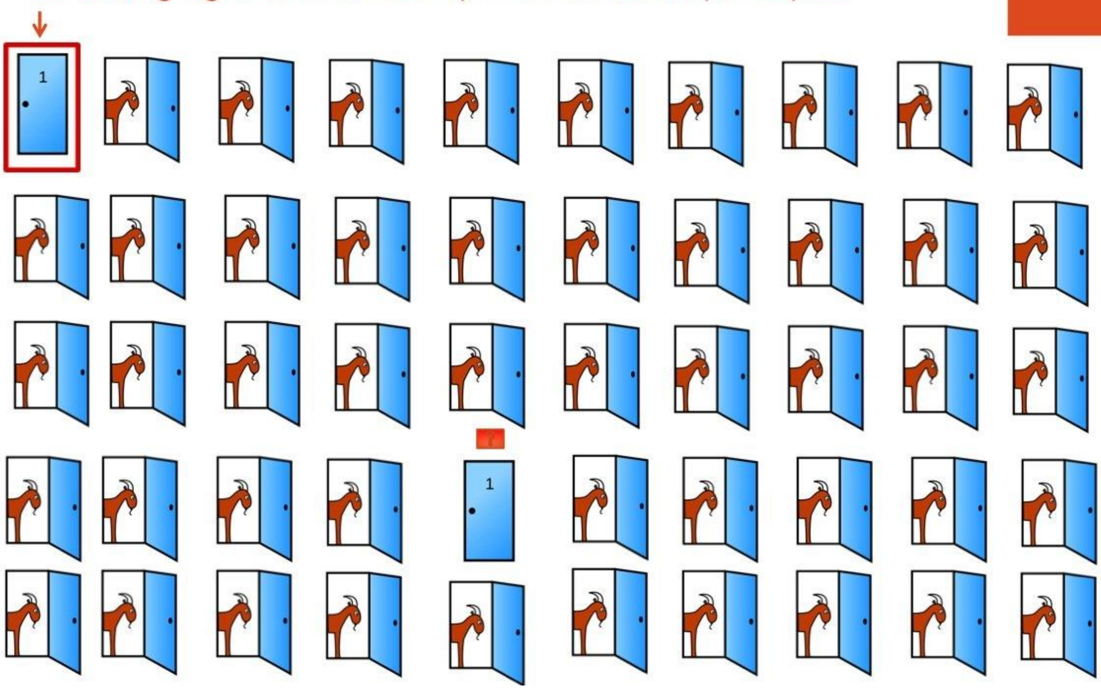
#1 The Monty Hall Problem

Wait, you still don't believe me?
Let's do the same game, but with 50 doors. You pick the first one.



#1 The Monty Hall Problem

And I show you 48 goats. Still feel so confident in your choice? Remember, you had a 1 in 50 chance of selecting right on the first try. It's the same principle.



Buffon's Needle

Introduction

This session will take more organization by the facilitator to help students find interesting results, find instructions as follows:

First, divide the class into groups. Have each group draw the following table on a sheet of paper: Also give each group the striped sheet shown below and a toothpick (students might want to colour their toothpick to make it easy to see which is theirs).

	Guess	Tick Mark	Total	Percentage
Crosses Line				
Student 1				
Student 2				
Student 3				
Doesn't Cross Line				
Student 1				
Student 2				
Student 3				

Each student should enter their guess as to the percentage of times a toothpick dropped on the striped sheet will cross a line. Then have each student drop the matchstick on the striped sheet **ten times**. They should record a tally in the *Tally* column for picks that cross and do not cross the line. Calculate totals and the overall percentage, ask learners to compare to their predictions.

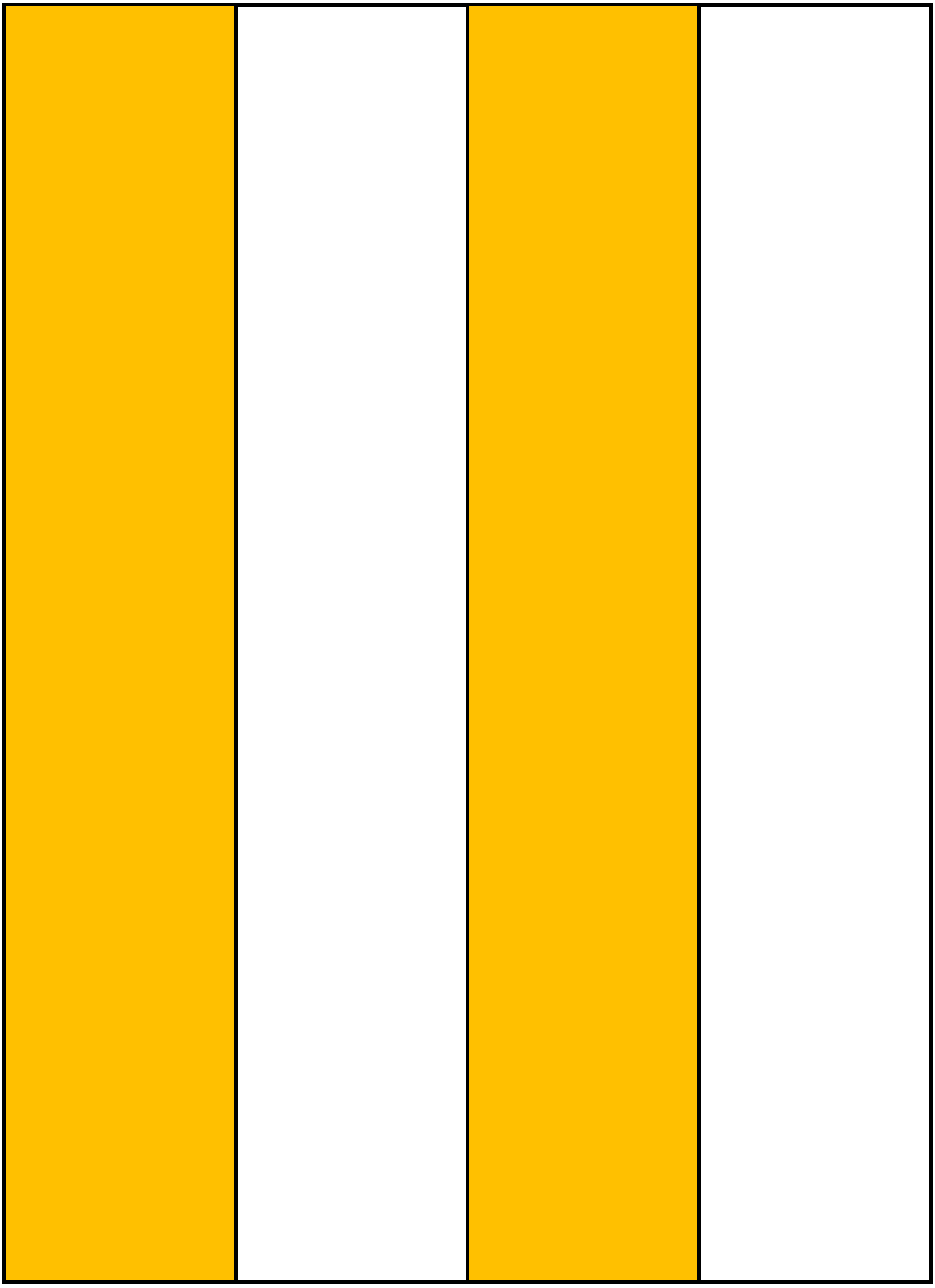
Next compute a total and percentages for each group of students and enter the numbers in the shaded cells. Finally, compute a total and percentages for the entire class. Measure the length of the matchsticks and compute $\frac{2L}{XP}$, where L is the length of the matchstick in mm, X is the width of the stripes in mm, and P is the Doesn't Cross Line percentage. What do you notice?

Solution

The answer should become closer to Pi π as you repeat the experiment more and more times!

Extension

Students can repeat the experiment more times or try with different length picks.



15

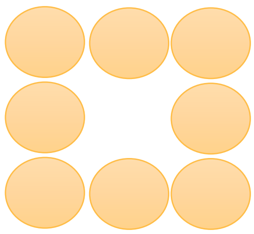
Introduction

15 is a game to challenge students to develop strategy and think mathematically. Many students will just try to guess where to place each number, perhaps randomly at first, but then should start to think strategically. Mental maths skills are used throughout the activity.

It may be useful for learners to use pieces of paper with numbers written on so that they can easily manipulate their answers, instead of writing them all on paper.

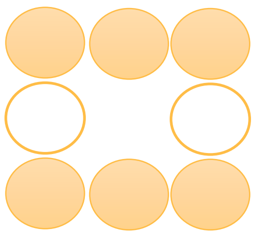
Solution

The secret here is to consider the total of all of the numbers used. We know that each number will be used once, so the overall total is 1+2+3+4+5+6+7+8 = 36



Total of all shaded = 36

We also know that the total of each of the rows and columns must be 15. If we just consider the rows



Total of two rows = 30

We therefore know that the value of the 2 circles in the middle must be 36 – 30 = 6
Once we place numbers totaling 6 in the centre there are multiple solutions which can be found, e.g..

3	5	7
4		**2**
8	1	6

7	2	6
5		**1**
3	3	9

Extension

The activity worked when the total in each row and column had to be 15. Will it work for any other totals? Can you explain why?

Addition Square

Introduction

The secret to this problem relies on the fact that different base number systems can be used in mathematics. Consider how when we record time, the next minute after 10:59 is 11:00. Minutes count up till 60 before we restart from 0. We call this *modular arithmetic*, in the case of time we operate with *modulo 60.*

Solution

The missing number is 10. This is an addition square modulo 12. This means that after doing the normal addition we take away as many multiples of 12 as we can before the number is less than 12.
So 11 + 11 = 22 and when we take away 12 we get 10.
Try it for the other additions to see that they follow this rule.

+	3	8	11
3	6	11	2
8	11	4	7
11	2	7	10

The second square is doing addition modulo 7. So the missing number is 10 because 5 + 5 = 10 and when we take away 7 from 10 we get 3.

+	1	3	5
1	2	4	6
3	4	6	1
5	6	1	3

Extension

Where do we use this type of mathematics in the real world? Can you create your own puzzles using modular arithmetic?

9 Dots

Introduction

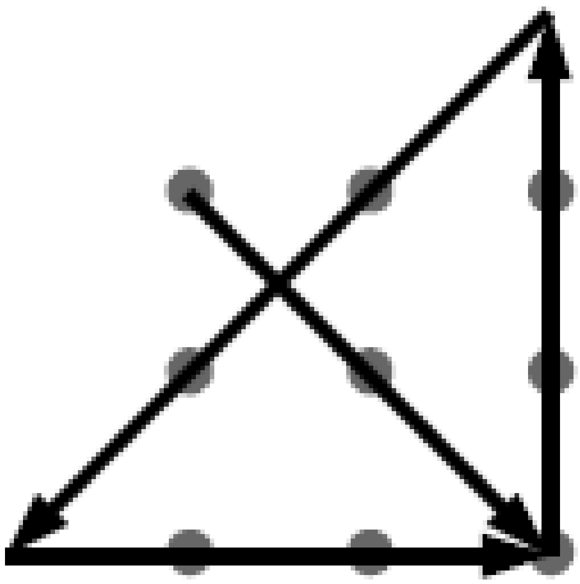
This problem is quite difficult and usually takes people a long time to solve. You have to think a bit differently to get the solution. Whether you find the solution or not, is not important. What's important is the thinking process you go through trying to solve the problem.

Often in school maths, you know what to do as soon as you see the problem. Here, you are faced with something new and the only thing you can do is try and make guesses. Then, you can improve on your guesses and try something different. Making many incorrect attempts is a good thing because you can understand why they don't work and improve on them.

We have included the solution here just as a check. But there is no value in just looking at the answer without trying to solve the problem first. The satisfaction of solving the puzzle on your own is much greater than someone telling you the solution.

Solution

The secret here is to pay close attention to what is and what is not stated in the question. There is nothing to stop your line extending outside of the shape like so:



Extension

You could try find the minimum number of lines it takes to draw through a 4 x 4 dotted grid

Find the path

Introduction

For this task students must be prepared to make several attempts before they get the solution. Remind students that getting an incorrect path is not bad. Students should be able to improve on incorrect paths and learn from them to get to the solutions.

Solution

<u>Can you make exactly 53?</u>
There is more than one way to make 53.
Here is one path:
5 + 7 + 6 + 6 + 9 + 4 + 5 + 7 + 4 = 53
Can you find any others?

	4	9	7	7	4	Finish
	8	9	4	5	7	
	6	6	4	9	9	
	7	8	8	8	6	
Start	5	5	6	5	5	

<u>Can you make exactly 60?</u>
One way to make 60 is:
5 + 5 + 8 + 6 + 9 + 9 + 7 + 7 + 4 = 60

	4	9	7	7	4	Finish
	8	9	4	5	7	
	6	6	4	9	9	
	7	8	8	8	6	
Start	5	5	6	5	5	

<u>What is the smallest possible number you can get?</u>
Students need to be systematic when creating paths and have a good system of recording their paths and their lengths.

The shortest path length is 48.
There are different ways to get 48.
Here is one way.
Can you find any others?

	4	9	7	7	4	Finish
	8	9	4	5	7	
	6	6	4	9	9	
	7	8	8	8	6	
Start	5	5	6	5	5	

Extension

Get students to create their own grids and challenges.

Consecutive Sums

Introduction

This problem can also be found on the Nrich website with additional notes: http://nrich.maths.org/507

The problem allows students to explore properties of numbers and start learning how to make *conjectures* (rules that they think might exist). You may wish to start the lesson by giving a challenge such as "The number 12 can be made by adding 3 consecutive numbers – 3+4+5. Can you make the number 14 by adding consecutive numbers?" (Solution = 2+3+4+5=14). Can you make all numbers this way? What numbers can be made in more than one way?

Solution

There is not a simple solution to this problem, but that is not the purpose. Hopefully learners will start to discover some patterns and general rules such as:

- You can make all odd numbers by adding 2 consecutive numbers (odd + even = odd)
- When you add 3 consecutive numbers you get the multiples of 3 (starting from 6)
- When you add 4 consecutive numbers you get all the even numbers which are not divisible by 4

2
3 = 1+2
4
5 = 2+3
6 = 1+2+3
7 = 3+4
8
9 = 4+5 = 2+3+4
10 = 1+2+3+4
11 = 5+6
12 = 3+4+5
13 = 6+7
14 = 2+3+4+5
15 = 7+8 = 4+5+6 = 1+2+3+4+5

You may also note that definitely not all numbers can be written as the sum of consecutive numbers, take for example the first 15 numbers written on the left:

The numbers 2,4 and 8 cannot be written as the sum of consecutive numbers.

Encouraging the class to collect results together in a table like this may help to see the pattern.

Extension

Can you think how we could probe algebraically some of these rules?

For example, take 2 consecutive numbers. If we call the first one **n** the second must be **n+1**
n+(n+1) = 2n+1
Since 2n is always even, the sum of 2 consecutive numbers (2n+1) must always be odd.

You could also try to get students to use their rules to predict whether larger numbers can be written as the sum of consecutive numbers, e.g. 315, 512, 406

IQ Challenge

Introduction

IQ challenges encourage people to think in different ways. The most important thing when doing these is to not let learners immediately give their answers to each other, instead let everybody attempt all the questions and then discuss with each other and the class later.

Solution

1. The answers and rules for the first question are as follows:

   24, 76: there are two sequences interwoven. Add 6 starting at 0 and deduct 6 starting at 100.

   64: add 1, 3, 5, 7, etc.

   19: add 1.5, 2.5, 3.5, 4.5, 5.5

   3281: $\times 5 + 1$ each time

2. The second question needs you to find a way that you can relate the three outer numbers to the inner number. If you add the two larger numbers and subtract the smaller, the answer is 3 times the middle number. Therefore missing number is 11:

   11: $[(23+15)-5] \div 3$

3. The third question challenges visualization and special awareness skills. Learners may find it easier to break the solution down into several parts. The answer is 50:

   50: 12 at the back, 10 at the sides, 6 underneath, 6 on top, 12 at the front, 4 on the insides.

Extension

Learners could design some of their own challenges to give each other. They could also try to grade the difficulty of their challenges as easy, medium or hard.

Counting Squares

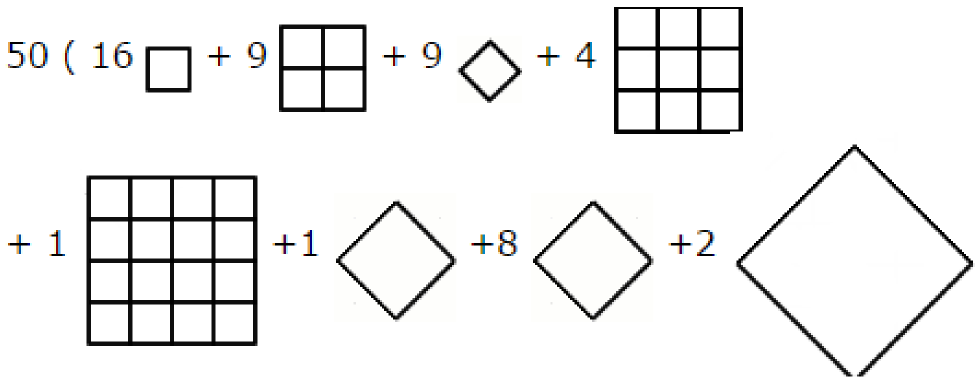
Introduction

This problem rewards those who can work systematically and devise a strategy to ensure that no squares are left out. You should let the learners at first try the problem and see if they get different answers, ask them to try and figure out why this is and explain their solutions to each other. This will be hard to do unless learners have come up with ways to record their findings. If learners are struggling you may want to get them to actually cut out different sized squares that they can manipulate on their board to assist with counting and record results in a table.

Solution

There are **50** squares, made up as follows:

Non-rotated: 16 one by one, 9 two by two, 4 three by three, 1 four by four
Rotated: 9 + 8 + 2 + 1 (varying sizes)



Extension

Instead of using a 5x5 dotted grid you can try other sizes and see if it is possible to generalize to nxn (note, there is a formula but it uses n^4 (n to the power 4) so it may not be immediately obvious. Below are the solutions for the first 6 grid sizes (from 2x2 up to 7x7)

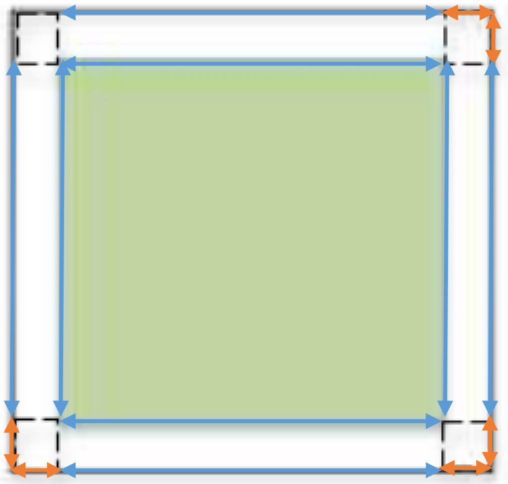
1, 6, 20, **50**, 105, 196

Fence Around a Field

Introduction

This problem probably appears more difficult than it is at first. Learners may wish to first try with an example using numbers for the length and width of the field, however this is not necessary to solve the problem.

Solution



This is a question about perimeter. The key to a quick and easy solution here is to understand that the edges of the fence parallel to the edges of the field will always be the same WHATEVER THE SIZE OF THE FIELD.

The extra fence is only needed at the corners (shown in the diagram).

In total there will be 8 extra pieces, all measuring the same length, perhaps called *w.*

We know that the extra length of fence is 1 metre which is the total length of the 8 pieces.
Therefore if 8w = 1m, **w = 12.5 centimetres**. The path is too narrow for a tractor but a mouse could easily run along it.

Extension

This problem sets up nicely for another called the *belt around the earth*, which goes as follows:

Imagine a piece of rope is tied the entire way around the earth, along the equator, so that it fits perfectly. If we instead wanted the rope to sit 1 metre above the earth at all points, how much longer does the rope have to be?

Hint: you do not need to know the size of the earth!

Number Puzzles 1

Introduction

Puzzles encourage people to think in different ways. These puzzles rely more on **thinking** than knowledge, so any learner can attempt them. There may be many right answers, and different approaches to the problems. Most important is not to let learners give their answers to each other, let everybody attempt all the questions and then discuss later.

The solutions are **not important**, it is about the thinking process learners go through in trying.

Solutions

1. The first two numbers add up to 11 so we can try different pairs. If they were 5 and 6 then the other number would have to be 12 or 13 but then none of the pairs add up to 22, so it's wrong. If we keep trying values we see that the first numbers can only be 3 and 8 and the other number will be 14 which works with all the information. However, if the numbers were bigger this would take much longer.

   A better way is to use **algebra** and **simultaneous equations**. Let the three numbers be x, y, and z. If the first pair is x and y, then we know that $x + y = 13$. If the second pair is x and z then, $x + z = 17$ and the last pair must be y and z so $y + z = 22$. This gives us three simultaneous equations:

   $$x + y = 11 \;\; ; \;\; x + z = 17 \;\; ; \;\; y + z = 22$$

   Equation 2 – Equation 1 gives: $z - y = 6$. Now add this to Equation 3 to get $2z = 28$. Which means $z = 14$. Using the original equations we can then work out that $x = 3$ and $y = 8$.

   Now try the question if the totals were 73, 140 and 109. Which method is faster now?

   This questions is all about being **systematic** - having a logical system to check all possibilities.

2.

3	1	2	1	3	2

   Start with the two 1s and work out what number can go between them. You will see that 3 is not possible.
   Harder:

4	1	3	1	2	4	3	2

   Do the same and you will see it's not possible to have a 2 or a 4 between the 1s. You can solve it from there.

3. $99 + \frac{9}{9} = 99 + 1 = 100$ or even $\frac{99}{.99} = 100$. This question requires some creativity!

Logic Puzzles 1

Introduction

Puzzles encourage people to think in different ways. These puzzles rely more on **thinking** than knowledge, so any learner can attempt them. There may be many right answers, and different approaches to the problems. Most important is not to let learners give their answers to each other, let everybody attempt all the questions and then discuss later.

Find the solutions is **not important**, it is about the thinking process learners go through in trying.

Solutions

1. First, fill the 3 litre jug so it is full. Pour the 3 litres into the 5 litre jug. Then fill the 3 litre jug again. Pour this into the 5 litre jug. When the 5 litre jug becomes full, you will have 1 litre left in the 3 litre jug.

2. Light both ends of the first rope and one end of the second rope. Once the first rope has burnt, light the other end of the second rope. Once the second rope has burnt, 45 minutes will have passed. Why? If you light both ends one of the first rope it will take 30 minutes to burn completely. After this, the second rope will need a further 30 minutes to burn, so if you light the other end as well, it will take 15 minutes to burn completely. Overall this will take 45 minutes.

3. First, fill the 5 litre bucket so it is full. Pour 3 litres into the 3 litre bucket. The 5 litre bucket now contains 2 litres. Empty the 3 litre bucket and pour the 2 litres into the empty 3 litre bucket. Now fill the 5 litre bucket. You now have 2 + 5 = 7 litres in the buckets! (There may be many different ways to do this, any is correct if you have followed the rules!)

Extension

What are all the different amounts we can measure using the 5 litre and 3 litre buckets?

What about for 4 litre and 2 litre buckets? What is the difference here?

Get students to come up with their own bucket and water puzzles. Make sure they check that they are possible! Try different size buckets or even add another bucket.

Geometry Puzzles 1

Introduction

Puzzles encourage people to think in different ways. These puzzles rely more on **thinking** than knowledge, so any learner can attempt them. There may be many right answers, and different approaches to the problems. Most important is not to let learners give their answers to each other, let everybody attempt all the questions and then discuss later.

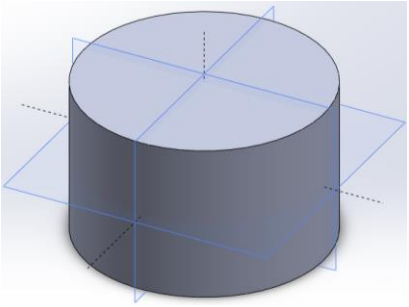
Find the solutions is **not important**, it is about the thinking process learners go through in trying.

Solutions

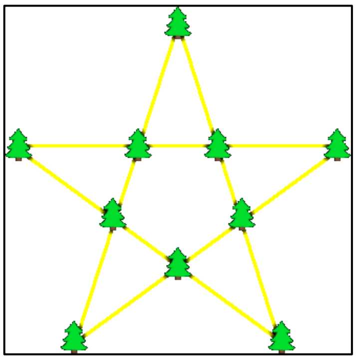
1. Let the radius of the Earth be R metres. Then the circumference, is $2\pi R$ metres. Which means the piece of string is also $2\pi R$ metres long. If we extend the string by 1 metre then the string is now $2\pi R + 1$ metres long. If we make a circle with this length then what is the new radius? If we make a circle with this string it means the circumference $= 2\pi R + 1$ metres. If the new radius is S then circumference $= 2\pi S$. But we also know circumference $= 2\pi R + 1$, the length of the string. So, $2\pi S = 2\pi R + 1$. We solve this for S. Divide both sides by 2π to get $S = R + \frac{1}{\pi}$.

   So S is $\frac{1}{\pi}$ metres longer than R. $\frac{1}{\pi}$ metres is about 0.16 metres, enough maybe for a cat to pass under! Surprisingly, you might have noticed we didn't need to know anything about the size of the Earth for this. What would the gap be if we did it with a football instead of the Earth?

2. First, cut the cake into 4 equal pieces with vertical cuts. Then make one horizontal cut in the middle of the cake to cut the four pieces into 8 equal pieces.



3. If we want 5 rows of 4 trees with only 10 trees, then each tree will have to appear in two different rows.



Special Numbers

Introduction

This problem comes from the Nrich website with additional notes: http://nrich.maths.org/2129

The problem allows students to explore properties of numbers and start learning how to make *conjectures* (rules that they think might exist).

Give the class some time to look for special numbers - once a few have been found, more will quickly follow. Once there is a consensus that a group of special numbers has been found, bring the class together and share what they think is going on. Ask them "Are you sure you've found all the special numbers? How could we be certain?" For this we need algebra.

Solution

Introduce the class to the algebraic representation of the problem.

Ask them, "If the first digit of the number is a and the second digit is b, what are expressions for the sum and product of the digits?" Students should work out it is $a + b$ for the **sum** and ab or axb for the **product**. If they don't see this, show them with some examples with numbers in place of a and b.

Next, how can we write the number as an algebraic expression? This is harder, you will probably need to show students that the number can be written as $10a + b$. For example, 35 can be written as $3 \times 10 + 5$, because 3 is the tens digit and 5 is the units digit.

Now we have an equation: $10a + b = (a + b) + (a \times b)$ (number = sum + product)

Subtract b from both sides and subtract a from both sides to get $9a = a \times b$. Divide by a because a is not zero and we get $b = 9$. This means any number with unit digit 9 will work. Try all 9 possibilities 19, 29, ..., 99 to check that it works.

If some students are not familiar with algebraic expressions, show them the extension questions.

Extension

Then present the suggestions for other types of special numbers:

- I add twice the tens digit to the units digit, then add this to the product of the digits. I get back to my original number.
- I add three times the tens digit to the units digit, then add this to the product of the digits. I get back to my original number.
- I add four times... or five times... or...

Can you work out which number are special using these definitions?

Can you use algebra to help you to find these special numbers?

Poisoned Wine Puzzle

Introduction

This puzzle is interesting because it is possible to use only 10 prisoners to find the bottle. You could start by asking what the king could do if he had 1000 prisoners to help students understand the puzzle. You could number the prisoners 0 -999, number the bottles 0 -999 and the prisoners drink a drop from the bottle of wine with their number on it. After 24 hours, one of the prisoners will die, and their number is the bottle number.

Then start to brainstorm ideas to do it with less prisoners.

The answer to this puzzle is to use binary numbers, so it could be given to students who have already seen binary numbers, or it could be done at the same time as Magic Cards puzzle and Russian Multiplication and then binary numbers discussed at the end.

Solution

First, let's line up our 10 prisoners and label them.

prisoner A	prisoner B	prisoner C	prisoner D	prisoner E	prisoner F	prisoner G	prisoner H	prisoner I	prisoner J
512	**256**	**128**	**64**	**32**	**16**	**8**	**4**	**2**	**1**

Also label the wine bottles 0–999 so we can tell them apart.

Label each bottle with both its decimal number and binary equivalent. **decimal=binary (the columns are 512,256,128,64,32,16,8,4,2,1)**	0	0000000000
	1	0000000001
	2	0000000010
	3	0000000011
	4	0000000100
	5	0000000101
	6	0000000110
	7	0000000111
	8	0000001000
	9	0000001001
	10	0000001010
	Etc.	

Now each bottle serves as a code describing which prisoners are to drink from it. In this system, a **one means the prisoner drinks** from it, a **zero means the prisoner doesn't.**

For example, bottle one should only be drunk by prisoner J since its binary is 0000000001. Whereas, Prisoners I and G should drink from bottle ten whose binary is 0000001010 because it has 1's in the columns that match up with prisoners I and G.

Continue this process until you have given out sips of wine from every bottle. After 24 hours, line up the prisoners in order and note which ones have been poisoned with ones and mark the rest with zeros. Convert this number back into decimal to reveal which bottle was poisoned.
If prisoners A,B,C,F and I die, which bottle was it in? (914)

Extension

If there were 2000 bottles, how many prisoners would you need?

Magic Cards

Introduction

Place the cards randomly on the desk and ask a student to find all the cards with their birth date on (e.g. they might be born on the 12th of November 2005 so they would be looking for all the cards with a 12 on it).

Ask them to tell you the colour of all the cards that their number is on.

Then all you need to do is add together the **top left number** of each of those cards and it will instantly tell you what date they were born on.

Do this a few times with the students. They can pick any new number for you to guess.

Then ask them to work out how you are doing it so quickly.

They should look for patterns on the cards.

The answer to this puzzle is to use binary numbers, so it could be given to students who have already seen binary numbers, or it could be done as a discovery of binary numbers at the same time as P15 Poisoned Wine puzzle and P17 Russian Multiplication and then binary numbers discussed at the end.

Solution

If you line up the cards in order from the biggest number in the top left to the smallest number in the top left, they are:

16,8,4,2,1

The numbers have been written on the cards so that if a number is on the card it means its binary equivalent has a 1 in that column.

e.g. the number 21 is 10101, so it would be on the 16 card, the 4 card and the 1 card.

Extension

Turn over the cards so that the students can't see them.

Can students recreate the cards themselves?

If you added one more card, how many more numbers could you do the trick with? Can you make up the new set of six cards?

Russian Multiplication

Introduction

This is an interesting method to do multiplications – involving only halving, doubling and adding.

Work through the 22 x 19 example in a traditional way and then the method described. Both ways should give the answer 418.

Then try some other multiplications, e.g. 25 x 37. Students should be keen to test their own multiplications and see if this process always seems to work.

Solution

It will work for any two numbers. An insight into why it always works can be given by trying multiplications where one of the numbers is a power of two, e.g. 30 x 36 or 30 x 64 or 30 x 128.

Extension

Understanding how Russian peasant multiplication is related to binary numbers
(taken from http://mathforum.org/dr.math/faq/faq.peasant.html)

Binary numbers are numbers written in base two instead of base ten. This means that place value depends on powers of two instead of powers of ten: instead of ones, tens, and hundreds places, base two has a ones place, a twos place, a fours place, and so on. For example, fourteen in base two is 1110:

1110 (base 2)
$= 1 * 2^3 + 1 * 2^2 + 1 * 2^1 + 0 * 2^0$
$= 8 + 4 + 2 + 0$
$= 14.$

Russian peasant multiplication is actually a quick way to convert two numbers to binary form, multiply them together, and convert back to our number system. The connection is not surprising, because binary numbers use base two, and Russian peasant multiplication depends on multiplying and dividing by two. To see the connection more clearly, let's investigate the problem 12*13.

Halving

You can convert a number to binary form by repeatedly dividing by two and keeping track of the remainders. Let's try 12:

12/2 = 6 remainder 0
6/2 = 3 remainder 0
3/2 = 1 remainder 1
1/2 = 0 remainder 1.

Reading the remainders from bottom to top, we get 1100, so 12 in base two is 1100.

Why does this conversion method work? Let's try cutting twelve in half again, the same way. This time, we'll write everything in base two. (Naturally, 2 in base two is 10.)

1100/10 = 110 remainder 0
110/10 = 11 remainder 0
11/10 = 1 remainder 1
1/10 = 0 remainder 1.

Dividing by two and then taking the remainder gives us a number's last digit in binary notation.

Here's what we know about 12, so far:

$12 = 1100$ (base 2)
$= 1*2^3 + 1*2^2 + 0*2 + 0*1$
$= 2^3 + 2^2$
$= 8 + 4$.

By halving 12 repeatedly, we have broken it down into powers of two.

The Distributive Property

We are trying to multiply 12 by 13. One way to do this would be to use long multiplication:

```
  13
 * 12
 ----
  26
+ 130
-----
 156
```

Notice that we are adding 2*13 and 10*13 to get our final answer. This works because of the distributive property:

$12 * 13$
$= (2 + 10) * 13$
$= 2*13 + 10*13$.

Of course, we can break 12 down any way we like, and still get the right answer. Let's use our previous work to split the problem into powers of two:

$12 * 13$
$= (4 + 8) * 13$
$= (2^2 + 2^3) * 13$
$= 2^2 * 13 + 2^3 * 13$.

If we can multiply 13 by 2^2 and 2^3, we will be finished.

Doubling

Repeatedly doubling a number multiplies it by powers of two. Let's try doubling 13:

Number	Multiplications so far	Power of 2
13	13	2^0
26	13*2	2^1
52	13*2*2	2^2
104	13*2*2*2	2^3

Our chart tells us that 2^2 * 13 + 2^3 * 13 = 52 + 104 = 156, so 12 * 13 = 156, and we are done.

Putting It All Together

We just used repeated halving and doubling to convert 12 to binary form, then multiply it by 13. Russian peasant multiplication does the same thing, but because it leaves out several steps, the process is much faster. Let's combine our doubling and halving steps to compare the two methods.

Number doubled	Multiplications so far	Power of 2	Number halved	Division Problem	Remainder
~~13~~	13	2^0	~~12~~	12/2 = 6	0
~~26~~	13*2	2^1	~~6~~	6/2 = 3	0
52	13*2*2	2^2	3	3/2 = 1	1
104	13*2*2*2	2^3	1	1/2 = 0	1

The columns used in Russian peasant multiplication are highlighted. Notice that when the number in the remainder column is 0, the corresponding row for Russian peasant multiplication is crossed out.

Langford’s Problem

Introduction

This puzzle can be done with pen and paper, or ideally it could be done with a pack of playing cards using the Aces, 2s, 3s, 4s and 5s. Students can then rearrange the cards to try and solve the three puzzles. The first puzzle is fairly easy, the second one is quite hard, and the third one is impossible! The proof that the third one is impossible is interesting because it uses colours – not all proofs involve algebra.

Solution

312132 (or the other way round – 231213)

41312432 (or the other way round – 23421314)

Adding in the 5s makes this impossible

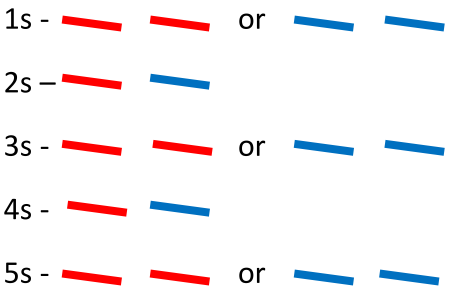
Imagine the ten numbers are written in this line:

The two 1s would have to be on the same colour lines to have one number between them.

The two 2s would have to be on different colour lines to have two numbers between them.

The two 3s would be on the same colour, 4s on different colours and 5s on the same colour.

So



There are five reds and five blues available in the line, but the best you can do with these constraints are 6 blues and 4 reds, or 6 reds and 4 blues. You cannot have five of each, so there is no way to fill in the numbers 1122334455.

Extension

Let n be the biggest number. So for n=3 and n=4 there is a solution, for n=5 there is no solution. For more information on which values of n in general have solutions see http://dialectrix.com/langford.html.

Coin game

Introduction

This simple coin game serves as an introduction to probabilities in tennis. It is really worth playing the game first with the students before trying to answer the questions to ensure everyone understands the way the game works, particularly when to stop playing.

Go through the answers to the first two questions before students try to fill in the table.

Watch students closely when they start filling in the table to ensure they are on the right track and help as necessary.

Solution

Which of the following scores are impossible in this game and why? 5-2, 7-4, 6-2,

5-2 is possible.

7-4 is impossible because that would be 11 tosses and you are only allowed 11.

6-2 is impossible because the game would have stopped before it got to this stage. Player 1 would have already won some time before.

Think of ways the game could end in a tie - Either 5-5, 6-4 or 4-6

What is the chance of winning this game?

The easiest way to fill in the table is to start in the top left corner, and each square is the sum of the number immediately to the left and immediately above. E.g. The numbers

				Player 1					
		0	1	2	3	4	5	6	7
	0	1	1	1	1				
	1	1	2	3	3	3			
Player 2	2	1	3	6	9	9	9		
	3	1	3	9	18	27	27	27	
	4		3	9	27	54	81	81	
	5			9	27	81	162		
	6				27	81			
	7								

It is tempting to add up the number of ways to win (1+3+9+27) and divide by the total of all the numbers in the table. But this would be wrong. It is wrong because the game lengths are different in each case.

The 1 in the blue outlined box represents a game of length 3, getting three heads in a row – the chance of which is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$.

The 3 is for a game where there was 4 heads and 1 tail. One way of this happening is THHHH. The probability of this is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$. Then there are the other two ways of HTHHH or HHTHH (note that HHHHT is not an option because the game would have stopped at 3-0). Therefore the probability of winning in this way is $3 \times \left(\frac{1}{2}\right)^5$

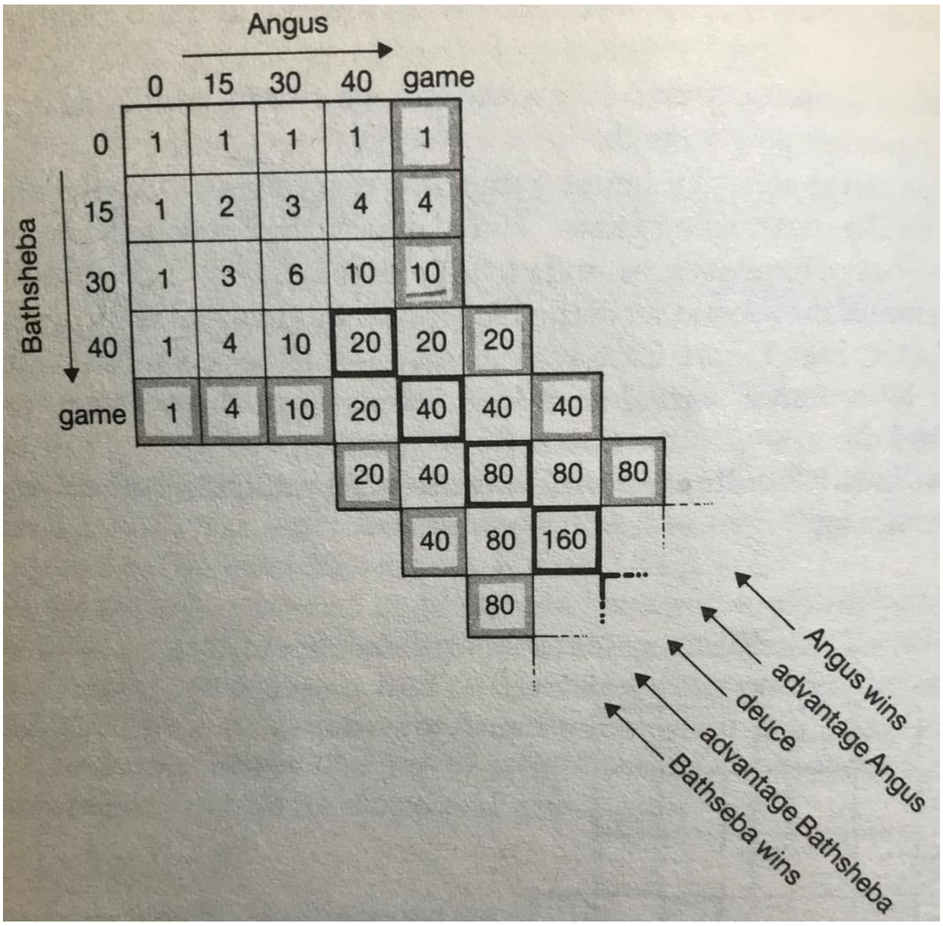
etc.

Therefore the probability of winning is $1 \times \left(\frac{1}{2}\right)^3 + 3 \times \left(\frac{1}{2}\right)^5 + 9 \times \left(\frac{1}{2}\right)^7 + 27 \times \left(\frac{1}{2}\right)^9 \approx 0.3418$

Extension

The way the scoring works in tennis is that you have to win by two clear points, so the same strategy as used above can be used to answer the question "If you have a one third chance of winning a point in tennis, what is your chance of winning a match?"

Here is the working for just winning one game in tennis with players called Angus and Bathsheba:



The maths involves summing an infinite series because tennis games could technically go on forever!

Domino Tilings

Introduction

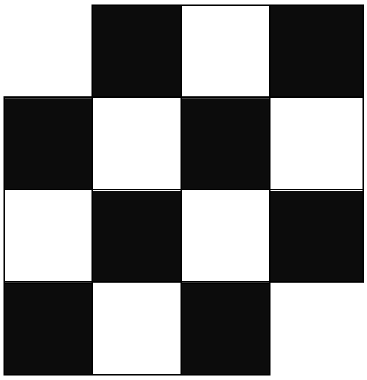
The idea of this exercise is to explore the possibility or otherwise of tiling (covering) certain shapes by dominos. While the question appears geometric, there the combinatorics of the situation also turns out to be important. One idea that learners should get from this exercise is that sometimes a construction is possible while other times it will not be; good students should also start to be able to make arguments (proofs!) for *why* a certain tiling is not possible. The proof of the most advanced example introduces a basic idea in combinatorial arguments, that or colouring.

Experimentation can be helped by the availability of squared paper, or paper and scissors to make the dominos!

Solution

a. tilings clearly exists (this could be done at the beginning together so that students get the idea).
Experimentation will suggest that in part
b. there is *no solution*, and good students will also spot the fact that the reason is that there is an *odd* number of squares. Indeed, removing one square as in part
c. will give a shape that can be tiled again. For part
d. experimentation quickly gives a tiling; for part
e., it can be shown by trying essentially all possibilities that there is *no solution* again.

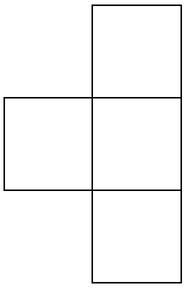
How so, since there is an even number of squares in both cases, and we removed two corners in each case? Well, the trick is to think of a chessboard colouring of our shape (it should be suggested to students to think of a chessboard, or explain to them what that looks like, but initially without further comment). The point is that no matter how one puts down a domino, it will always cover *one* black and *one* white square. So in part e. no tiling can exist since the coloured version has a different number of black and white squares:



This argument will also show that no tiling in case f. exists either; this case would be impossible to do by trial and error.

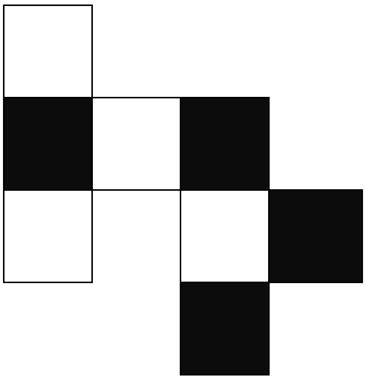
Extension

Students can go on to discuss which shapes can be tiled, and which cannot. For example,



clearly cannot be tiled; this is very easy to see directly, but the colouring argument also proves it. A more interesting question is whether there are any shapes that cannot be tiled, even though they contain the same number of black and white squares. This could be set as a challenge to a good class. A good student might find the following example:

(placing this "shape" on a chessboard so that one square becomes white, the other black). While in some sense this is "cheating", this would be a good point to talk about *connectedness* of our shapes. A more interesting example, "doubling up" the previous example with 4 squares, is



While this shape has 4 black and 4 white squares, it still cannot be tiled, as a simple experimentation shows. Further references:

https://en.wikipedia.org/wiki/Mutilated_chessboard_problem

https://en.wikipedia.org/wiki/Domino_tiling

https://arxiv.org/pdf/math/0501170.pdf

Collatz conjecture

Introduction

The activity is interesting because it is easy to understand the rules and start generating sequences, but it is an unsolved problem in mathematics whether all number end up at one. Many people have tried to find a sequence that doesn't go to one, or to prove that all sequences go to one and no one has succeeded. There is a prize of at least $2000 dollars if you succeed!

Solution

The shortest sequences are going to involve powers of 2, e.g. 1,2,4,8,16,32 etc. as these numbers "snowball" to 1. The longest sequence for a number smaller than 100 is for 97, which takes 118 steps but finally gets to 1. The number just before, 96, only takes 12 steps!

Extension

A computer or tablet would be very useful to try and find very long sequences. Here are some instructions for GeoGebra, which you can find online (https://www.geogebra.org/classic/spreadsheet) or it may be installed on the tablet and you do not need the internet.

You will need to use the Mod command:

Mod[<Dividend Number>, <Divisor Number>]

to check if the number is even. Mod is short for Modulo and gives the remainder when you do a division. e.g. Mod[10,3] would be 1 because there is 1 leftover when you divide 10 by 3.

You will also need the If command:

If[<Condition>, <Then>, <Else>]

to choose what to do if it is even and what to do if it is odd.

Put together, here is the formula you should put into cell A2, once you have put a starting number in A1.

=If[Mod[A1, 2] == 0, A1 / 2, A1*3 + 1]

Then just hover your mouse pointer in the bottom right corner and click the left button and hold down and drag down lots of cells. You should see the sequence appear.

If you want to be super clever you could try and combine two If statements so that if the cell was equal to 1 it would stop calculating and just say "STOP".

Apple Teaser

Introduction

This session will require the students to read and interpret all the information provided to successfully work out the total number of apples in the three rooms. The students can work individually or in small groups. The instructions are as follows:

Have the students draw the following table on a sheet of paper:

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27

Each number in the table represents the total number of apples, $9\ apples\ in\ each\ room \times 3\ rooms = 27$

Get the students to work out what the minimum and maximum possible numbers of apples are and cross out the numbers outside of this range. Also, the students should think about the answers to Adam and Belle's questions and cross out any other numbers.

Which numbers are left? Circle them. For each of these numbers, can you work out how many apples there might be in each room?

Solution

Working out the minimum number of apples: We know that there is at least one apple in each room and each room has a different number of apples. So, the minimum number of apples is $1 + 2 + 3 = 6$. Cross out numbers $1, 2, 3, 4, 5$.

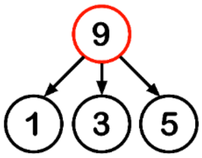
Working out the maximum number of apples: We also know that no room has more than nine apples, so the maximum number of apples is $7 + 8 + 9 = 24$. Cross out numbers $25, 26, 27$.

From the questions, we know that the total number of apples is not an even number and it is not a prime number. So, cross out the even numbers and prime numbers.

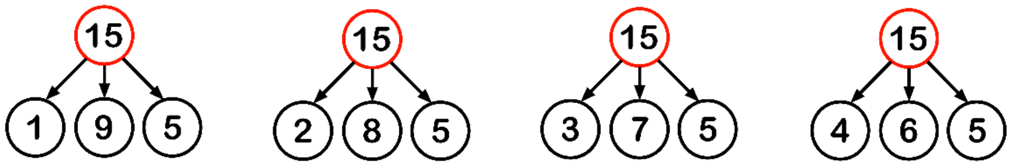
Circle the numbers that remain.

~~1~~	~~2~~	~~3~~	~~4~~	~~5~~	~~6~~	~~7~~	~~8~~	9
~~10~~	~~11~~	~~12~~	~~13~~	~~14~~	15	~~16~~	~~17~~	~~18~~
~~19~~	~~20~~	21	~~22~~	~~23~~	~~24~~	~~25~~	~~26~~	~~27~~

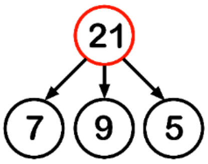
Case 1: If the total number of apples is 9 and you have 5 of them in your room, then Adam and Belle have 4 apples between them. So, one has a single apple and the other has 3 apples.



Case 2: If the total number of apples is 15 and you have 5 of them in your room, then Adam and Belle have 10 apples between them. Here, there are many possible combinations:



Case 3: If the total number of apples is 21 and you have 5 of them in your room, then Adam and Belle have 16 apples between them. So, one has 9 apples and the other has 7 apples.



So, what is the question you should ask?

You should ask: *"Is the total number of apples 15?"* and reconsider each case.

Case 1: The total number of apples is 9: then the person who has the single apple knows that the total cannot be greater than $1 + 8 + 9 = 18$ and the person who has 3 apples knows that the total cannot be greater than $3 + 8 + 9 = 20$. Either Adam or Belle should tell Ruben that the answer is 9.

Case 2: The total number of apples is 15: then Ruben will answer "Yes" and any one of you can tell him that the answer is 15.

Case 3: The total number of apples is 21: then the person who has 9 apples knows that the total cannot be less than $9 + 1 + 2 = 12$ and the person who has 7 apples knows that the total cannot be less than $7 + 1 + 2 = 10$. Either Adam or Belle should tell Ruben that the answer is 21.

Circumference

Introduction

The answers to the two questions are related and can be done in any order. Question 1 uses numerical values to calculate the circumference but question 2 requires students to leave the width as a constant d.

These puzzles are interesting because the result is not immediately obvious. It will get students thinking about how their answers apply to different circles, regardless of their size. Encourage group discussions about the answers that they found. Students can leave their answers in terms of π or use the approximation $\pi \approx 3$.

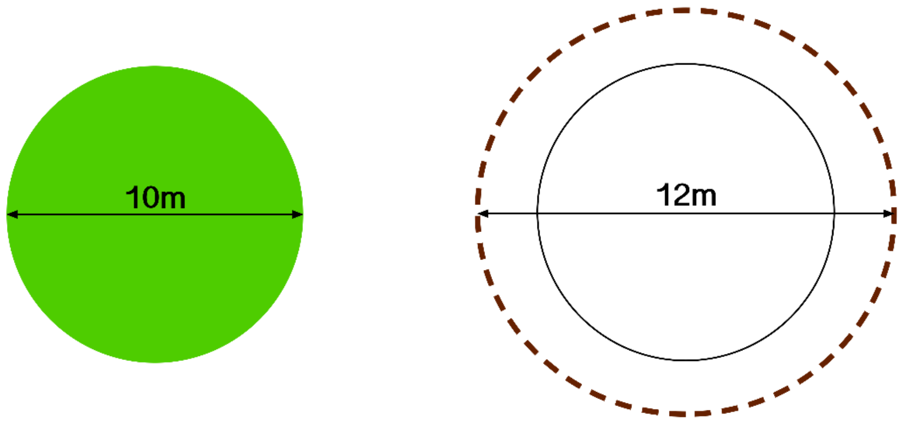
Circumference of a circle $= diameter \times \pi$

Solution

Question 1

Firstly, you need to work out how much fencing was needed **before** the extension. To do this, students will need to calculate the circumference of the original field $= width \times \pi = 10\pi$

Secondly, you need to work out how much fencing will be needed **after** the extension. The width of the bigger field is now $1 + 10 + 1 = 12$. Then the circumference of the bigger field $= width \times \pi = 12\pi$



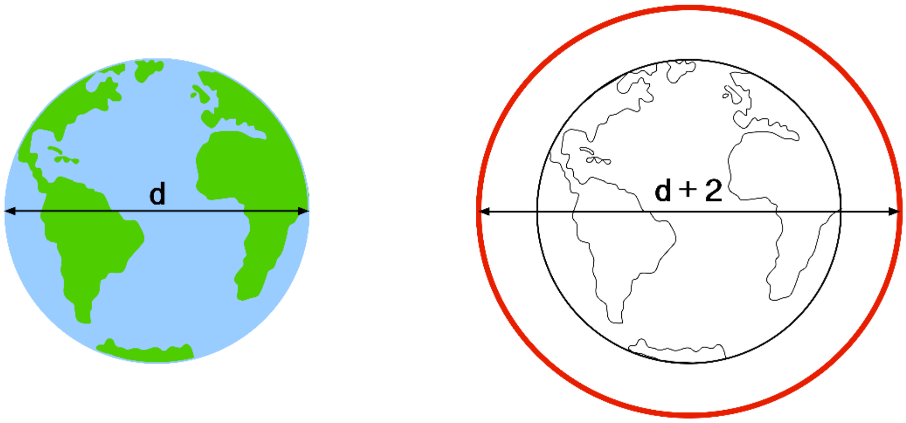
The amount of fencing the farmer would need is the difference between the two circumferences worked out above. The farmer would need $12\pi - 10\pi = 2\pi$ more fencing. This is approximately $\approx 2 \times 3 = 6$ metres.

Question 2

The method is the exact same as for question 1 except here you leave the width as a constant d.

Firstly, you need to work out how much rope was needed when it was tied around the equator of the earth. To do this, students need to calculate the circumference of the earth $= width \times \pi = d\pi$

Secondly, you need to work out how much rope will be needed if we instead want the rope to be 1 metre above the earth at all points. The width of the bigger circle is now $1 + d + 1 = d + 2$. Then the circumference of the bigger circle $= width \times \pi = (d + 2) \times \pi = d\pi + 2\pi$.

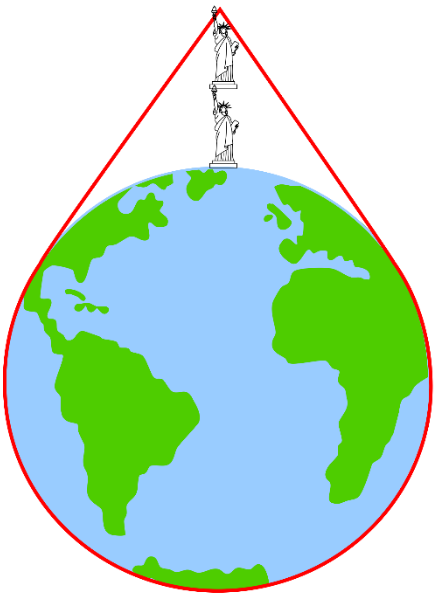


Like how this was worked out in question 1, the amount of extra rope needed is the difference between the circumference of the earth, $d\pi$, and the circumference of the bigger circle, $d\pi + 2\pi$.
Amount of rope $= (d\pi + 2\pi) - d\pi = 2\pi$.

What do you notice about the answers to question 1 and question 2? What would happen if you did the same with a smaller planet? Why do you think this happens?

Extension – something to think about, not work out!

What is even more interesting is what happens when the rope is lifted by a single point. The rope around the bottom of the earth is taut but now there is a large clearance at the top. For an additional 2 metres length of rope, the distance from the earth's surface to the peak of the rope would be enough to fit **two Statue of Liberty's underneath it!**



Are you surprised that such a small increase in the length of the rope will reach such a large height? Think about how this height would change when you change the size of the ball.

Matchstick Puzzles

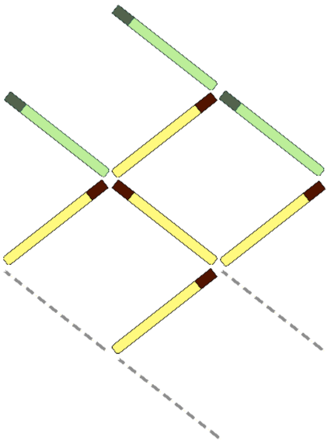
Introduction

This is an excellent activity to develop thinking skills. The four puzzles get more challenging and each require the students to think about how they can use the matchsticks in different ways. Don't peek at the solutions right away, give yourself time to figure them out!

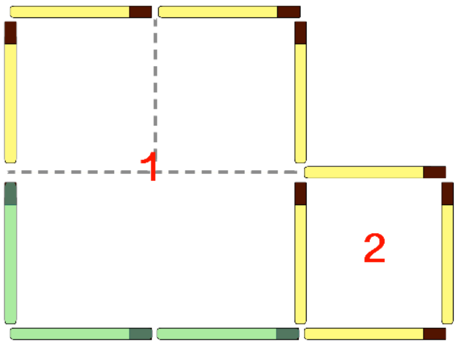
Students can approach this activity by drawing the matchsticks on some paper, or they can make the configurations with cocktail sticks or pencils laid out on the desk and try moving them around instead.

Solution

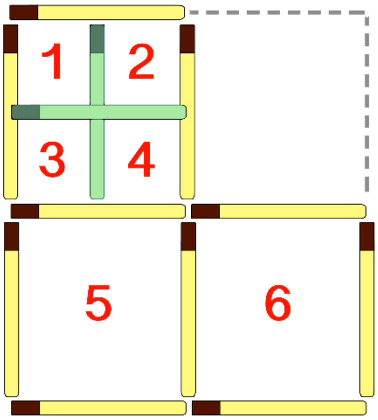
Question 1)



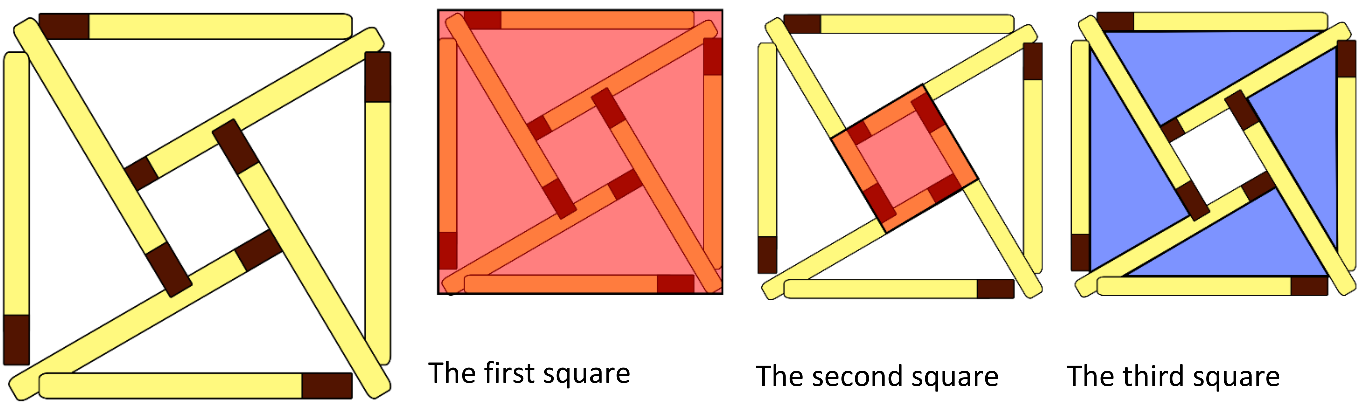
Question 2)



Question 3)



Question 4)



The first square

The second square

The third square

Gabriel's Problem

Introduction

This problem will get students to think methodically and will get them to develop a strategy for considering each of the different cases. You should let learners work out the factors of each row and column before assisting them with considering each of the cases.

Get students to start by drawing this grid:

			24
			40
			378
60	21	288	

Solution

Notice that there is only one combination of factors that multiply together to get 21, so we start here. In fact, $21 = 1 \times 3 \times 7$. We know, that the middle column therefore has the numbers 1, 3 and 7. Consider these three numbers as factors of each row:

- 24 is divisible by 1 and 3
- 40 is divisible by 1
- 378 is divisible by 1 and 7

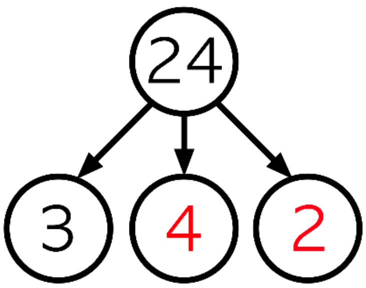
40 is not divisible by 3 or 7 and so 1 should go in the middle box. Then that leaves 3 to go in the top box and 7 in the bottom box.

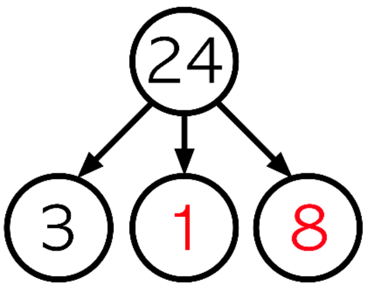
	3		24
	1		40
	7		378
60	21	288	

Now we can look at each row in turn to find the possible factors.

<u>Row 1</u>

We know that the numbers in this row must multiply together to get 24. The number 3 is one of those numbers. So, consider what other two numbers could be:

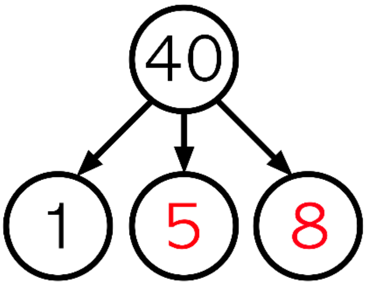




Notice that 1 has already been used in the second row of the grid, leaving us with the first combination of factors. Therefore, 4 and 2 go in the remaining two boxes in the first row.

Row 2

Again, we know that the numbers in this row must multiply together to get 40. The number 1 is one of those numbers. So, the other two numbers are:



Therefore, 5 and 8 go in the remaining two boxes in the second row.

Row 3

We take a different approach to this row than we did for the others. Out of the possible numbers 1-9, we have already used $1, 2, 3, 4, 5, 7$ and 8. Which means that 6 and 9 are the only two numbers left and therefore must go in the remaining two boxes in the final row.

We need to work out the order now.

Step 1. Consider the bottom left box, this is either 6 or 9. We know that 60 is divisible by 6 but is not divisible by 9. Therefore, the order of the bottom row must be:	Step 2. Consider the middle box on the left, this is either 8 or 5. We know that 60 is divisible by 5 but is not divisible 8. Therefore, the order of the bottom row must be:	Step 3. Then we can fill in the top row as follows:

Step 1:

	3		24
	1		40
6	7	9	378
60	21	288	

Step 2:

	3		24
5	1	8	40
6	7	9	378
60	21	288	

Step 3:

2	3	4	24
5	1	8	40
6	7	9	378
60	21	288	

Check for yourself that this is correct.

Extension

Can you also fill out these grids? One of them has more than one solution:

			24
			120
			126
24	72	210	

			28
			144
			90
40	48		

For this grid, Gabriel used the numbers 1, 2, 3, 4, 5, 6, 9, 10 and 12.

			12
			60
20	135		

Gabriel used the numbers 1, 2, 3, 4, 6, 8, 9, 12 and 16 to make this grid. How many solutions can you find? Is this grid easier or harder to complete the ones above? Why do you think this is the case?

			36
			384
			144
96	384	54	

You can also make your own 3x3 grids and swap them with friends. Try to solve each other's grids!

River Crossing

Introduction

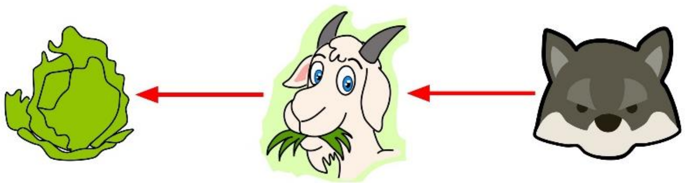
River crossing puzzles are challenging because it involves the students having to think logically about the order of movement and the rules on the river banks. The problem often requires students to use trial and error by playing out different scenarios until they either reach an endless loop, a dead-end or an acceptable solution.

A fun way to solve this problem is to work in groups with each person pretending to be either the farmer, the wolf, the goat or the cabbage. Students should draw a line on the floor that illustrates the river and then physically act out the problem by crossing the line.

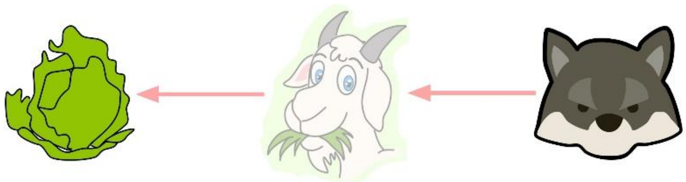
There is a more rigorous approach to this problem that will be introduced in the extension.

Solution

All solutions will start with moving the goat first. Can you think why this is the case? It is because the wolf eats the goat and the goat eats the cabbage, shown by the red arrows below:



We need to break this chain. Removing one of the three objects needs to leave the remaining two objects with no arrows between them, that way nothing will get eaten when they are left on their own.

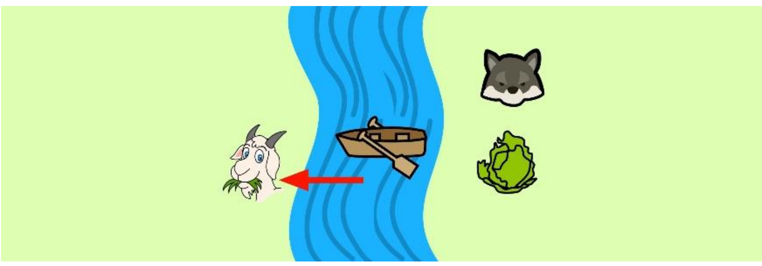


Therefore, the first move will always be to move the goat to the other side.

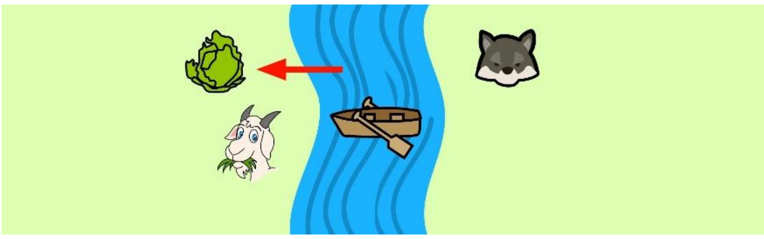
There are two possible solutions to this puzzle and they are explained below:

<u>Solution 1</u>

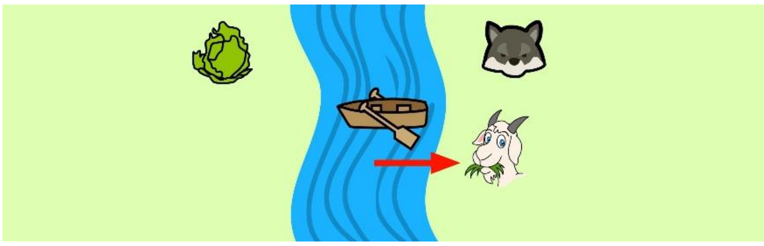
At the start the wolf, the goat and the cabbage are all together with the farmer. As we discussed above, the farmer must take the goat to the other side first and then he travels back alone.



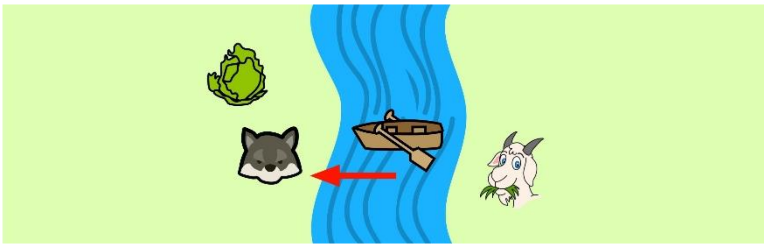
Would you now move the wolf or the cabbage? In this case the farmer will take the cabbage to the other side.



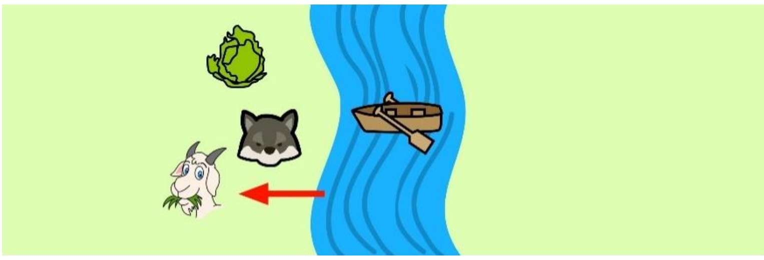
But now the goat and the cabbage are on the same side and cannot be left alone because the goat will eat the cabbage. So, the farmer must take either the goat or the cabbage back with him. If he chose to take the cabbage back with him then the farmer would end up in an endless loop of moving the cabbage. So, he has no choice but to take the goat back with him.



The goat and the wolf are now on the same side of the river and cannot be left alone because the wolf will eat the goat. There is no point taking the goat back because then the farmer will end up in another endless loop of moving the goat, so he must take the wolf over to the other side.



The wolf and cabbage can be left alone on the same side of the river and nothing will get eaten. So, the farmer can travel back alone to pick up the goat and return to the other side of the river.



Now the farmer, the wolf, the goat and the cabbage have all successfully crossed the river with nothing getting eaten.

This process can be written more simply:

(1) Move the goat to the other side
(2) Move the cabbage to the other side
(3) Move the goat back
(4) Move the wolf to the other side
(5) Move the goat to the other side

<u>Solution 2</u>

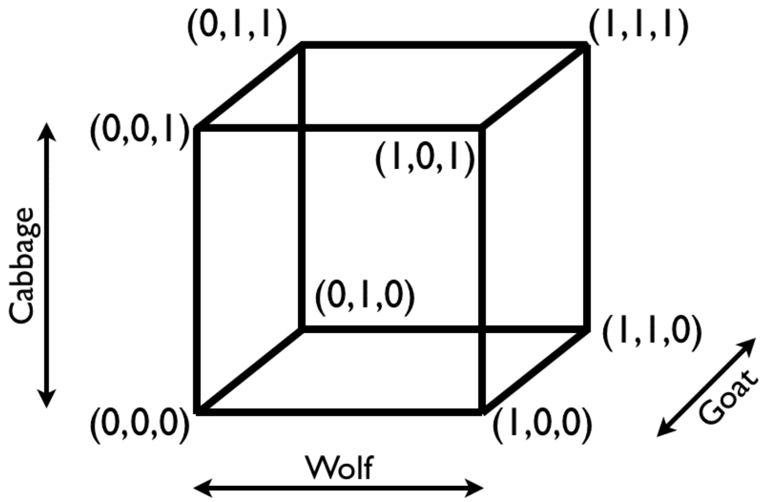
Another alternative method moves the wolf instead of the cabbage in step (2):

(1) Move the goat to the other side
(2) Move the wolf to the other side
(3) Move the goat back
(4) Move the cabbage to the other side
(5) Move the goat to the other side

Extension

How could you represent the problem in 3D?

Think about the space where the axes are represented as the wolf, the goat and the cabbage, like this:



How does this change the problem? The brackets represent the river bank and the three numbers inside represent the wolf, the cabbage and the goat respectively. The number 1 means that they are on the other side of the river bank and the 0 means that they aren't. For example:

- $(0,0,0)$ is where we started, with all three on the starting side of the river.
- $(1,0,1)$ is where the wolf and the cabbage have crossed the river.
- $(1,1,1)$ is what we want to get to, with all three on the other side of the river.

The problem is now to get from the $(0,0,0)$ vertex of the cube to the $(1,1,1)$ vertex, using the edges of the cube as a movement of either the wolf, the goat or the cabbage.

How would you do this? Hint: think about which edges represent a movement that results in something getting eaten.

Can you recreate the two solutions we found above on this cube?

Coconut Trader

Introduction

The key to this question is to realise that you should take all the coconuts from one sack until that sack is empty. Then do the same with the second sack. If each time you reach a checkpoint you take one coconut from each sack, then you would have no coconuts left by the time you reach the market! Leave students to work this out for themselves but if they get stuck then give them a hint.

Solution

Emptying the first sack

You currently have three sacks so at each checkpoint you must give **three** coconuts each time. That means that the sack will be empty after $30 \div 3 = 10$ checkpoints. You throw away this empty sack.

Emptying the second sack

You now have two sacks left so at each checkpoint you must give **two** coconuts each time. That means that the sack will be empty after $30 \div 2 = 15$ checkpoints. You throw away this empty sack.

Emptying the third sack

You now have one sack left so at each checkpoint you must give **one** coconut each time. You have already passed $10 + 5 = 15$ checkpoints, which means you have five checkpoints left. That means you must give out $5 \times 1 = 5$ coconuts from your final bag.

Therefore, you are left with $30 - 5 = 25$ coconuts to sell at the market.

Extension

Here are two more puzzles to solve:

1. Cars across the desert

A car must carry an important person across the desert.

There is no petrol station in the desert and the car has space only for enough petrol to get it half way across the desert.

There are also other identical cars that can transfer their petrol into one another.

How can we get this important person across the desert?

Handshake Puzzles

Introduction

This can be done by students thinking about it in their heads or they can arrange themselves into groups of $1, 2, 3$ etc … to act out the different scenarios and count the number of handshakes they make.

For this puzzle we will assume that a handshake is shared i.e.. that if person A shakes the hand of person B, then person B has already shaken the hand of person A. The handshake should not be counted twice.

Each stage gets progressively more difficult. The extension section increases the number of people even further with the hope that by considering more cases, students will start to look for a pattern.

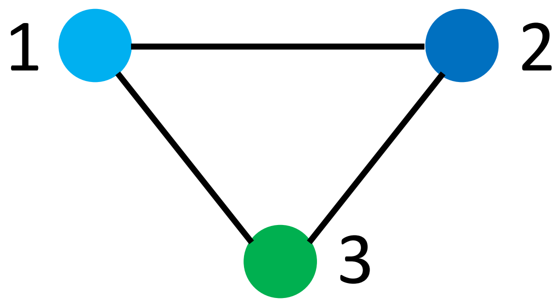
Solution

Stage 1

There are only two people at a party. Once they have shaken hands with each other there is no one left at the party to shake hands with. The total number of handshakes is 1.

2 people, 1 handshake.

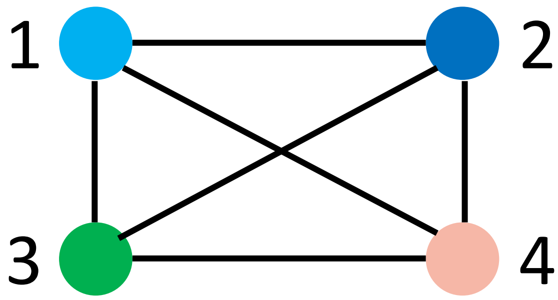
Stage 2



There are three people at a party. The first person shakes hands with the other two people. The second person has already shaken hands with the first person, but he can still shake hands with the third person. Person three has already shaken hands with everybody at the party, so there are no additional handshakes. So, the total number of handshakes is $2 + 1 + 0 = 3$.

3 people, 3 handshakes.

Stage 3



There are four people at a party. The first person can shake hands with three people, the second person can shake hands with two additional people and the third person can shake hands with one additional person. Person four has already shaken hands with everybody at the party, so they have no additional handshakes. So, the total number of handshakes is $3 + 2 + 1 + 0 = 6$.

4 people, 6 handshakes.

Extension

Ask the students to record their results in a table and fill in values for up to 8 people.
Can you spot a pattern in the numbers? Can you find a general rule for n people at a party? Use this to work out how many handshakes there would be if there were 30 people attending the party.

Treasure Hunt

Introduction

There are two different approaches to this puzzle. You can assume that:

- Each chest contains the treasure, or that
- Each statement is true

and consider the four cases that arise.

The way a student structures their answer will play a big part in how easy they find the answer.

Solution

<u>Assumption 1</u>

The treasure is in chest A. That means that statement A must be the telling the truth. That also means that statement B is telling the truth. Since the treasure is in chest A, the treasure cannot be in chest C and therefore statement C is telling the truth.

The treasure is in…	**A "The treasure is in here"**	**B "The treasure is in chest A or chest D"**	**C "The treasure is not in here"**	**D "The treasure is in here"**
Chest A	✓	✓	✓	✗

There are three chests that are supposedly telling the truth, but there should only be one true statement. So the treasure cannot be in chest A.

<u>Assumption 2</u>

The treasure is in chest C. That means that statement C must be lying. The treasure is not in chest A or D, making those statements lies as well. Meaning that statement B is also a lie.

The treasure is in…	**A "The treasure is in here"**	**B "The treasure is in chest A or chest D"**	**C "The treasure is not in here"**	**D "The treasure is in here"**
Chest C	✗	✗	✗	✗

None of the chests are telling the truth, but there should be one true statement. So the treasure cannot be in chest C.

Assumption 3

The treasure is in chest D. That means that statement B and D are telling the truth. Statement A must be lying since the treasure is not in chest A. Since the treasure is not in chest C, statement C must be telling the truth.

The treasure is in…	**A "The treasure is in here"**	**B "The treasure is in chest A or chest D"**	**C "The treasure is not in here"**	**D "The treasure is in here"**
Chest D	✗	✓	✓	✓

There are three chests that are supposedly telling the truth, but there should only be one true statement. So the treasure cannot be in chest D.

Assumption 4

The treasure is in chest B. That means that the treasure is not in A or D, making those statements a lie. Now we know the treasure is not in chest A or D, then statement B must also be a lie. Since the treasure is not in chest C, this statement is true.

The treasure is in…	**A "The treasure is in here"**	**B "The treasure is in chest A or chest D"**	**C "The treasure is not in here"**	**D "The treasure is in here"**
Chest B	✗	✗	✓	✗

Only one statement is telling the truth. **This means that the treasure is in chest B.**

Two Eggs

Introduction

This mind-teaser is an interesting one because there are so many different approaches that will allow you to find out the answer, but there is only one optimal approach! Students will develop their critical thinking skills by using trial and error and thinking about how to improve their method each time.

There are no tricks involved with this puzzle. There is no need to consider terminal velocity, potential energy or wind resistance. Both eggs are identical.

Get students to think about the puzzle themselves before reading on.

Solution

<u>Approach A - worst case 100</u>

The first logical approach would be to test every floor from the bottom floor to the top floor. So, we could start from the 1st floor and drop the egg from there. If it survives we can move up to try again from the 2nd floor, then the 3rd floor and so on. We keep moving a floor up each time until the egg breaks, and we get the solution. For example, if the egg breaks on the 67th floor, we know that the highest floor than an egg can withstand a drop is from the 66th floor. In this case, we would have dropped a single egg 67 times!

There is also a chance that the egg will not break from the 100th floor and by using this approach we would have dropped the egg 100 times in total to find this out!

This approach is clearly not optimal.

<u>Approach B - worst case 50</u>

Can we can halve the number of egg drops? What if we start by dropping the first egg from the 50th floor?

What happens if the egg breaks from the 50th floor?

If the egg breaks from being dropped from the 50th floor, then it would break from every floor above this. We would then only have 1 egg remaining to check all the floors below in a similar method to Approach A (one floor at a time, starting from the 1st floor to the 49th floor). As a worst case, we would have to check 50 floors.

What happens if the egg does not break from the 50th floor?

If the egg did not break from being dropped from the 50th floor, then it would not break from any floor below this. So again, we would only have 1 egg remaining to check all the floors above in a similar method to Approach A (one floor at a time, starting from the 51st floor to the 100th floor). As a worst case, we would have to check 50 floors in total.

This is still not optimal.

<u>Approach C - worst case 19</u>

What would happen if we started off dropping our first egg at intervals of 10 floors? (for example, 10th floor, 20th floor, 30th floor etc.).

We would start by dropping the egg from the 10th floor. If the egg does not break, then it would not break from any floor below this and we can try again from the 20th floor. If the egg still does not break, we can try from the 30th floor, then the 40th floor and so on. Once the egg breaks from being dropped, we know that the solution is somewhere in the 9 floors below.

For example, the egg did not break from the 90th floor but it did break from the 100th floor, so we know that the egg must break on one of the floors between 90 and 100. We test the 91st floor first, then the 92nd floor and so on until the egg breaks and we have the solution.

The worst case would be if the egg breaks from the 99th floor because this would take us 19 egg drops in total.

1st egg breaks from floor…	**Test the 2nd egg from floor…**	**Worst-case total drops**
100	91 ⟶ 92 ⟶ 93 ⟶ 94 ⟶ 95 ⟶ 96 ⟶ 97 ⟶ 98 ⟶ 99	10 + 9 = 19

Why are these not the best approaches? How can you make them optimal in their worst case?

<u>The optimal solution - worst case 14</u>

We should start by dropping an egg from the 14th floor. Can you think about why before reading on?

What happens if the egg breaks from the 14th floor?

If the egg breaks from being dropped from the 14th floor, then it would break from every floor above this. We should then check the 1st floor, then the 2nd floor, all the way up to the 13th floor (13 floors). In the worst-case scenario, we would have to check 14 floors in total: 14, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13.

1st egg breaks from floor…	**Test the 2nd egg from floor…**	**Worst-case total drops**
14	1 ⟶ 2 ⟶ 3 ⟶ 4 ⟶ 5 ⟶ 6 ⟶ 7 ⟶ 8 ⟶ 9 ⟶ 10 ⟶ 11 ⟶ 12 ⟶ 13	1 + 13 = 14

What happens if the egg does not break from the 14th floor?

If the egg did not break from being dropped from the 14th floor, then it would not break from any floor below this. Then we would check the 27th floor next. This is because if it breaks at the 27th floor, we would only have to check all the floors from the 15th to the 26th one (12 floors). Start by checking the 15th floor, then the 16th floor, all the way up to the 26th floor. In the worst-case scenario, we would again have to check 14 floors in total: 14, 27, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26.

1st egg breaks from floor…	**Test the 2nd egg from floor…**	**Worst-case total drops**
27	15 ⟶ 16 ⟶ 17 ⟶ 18 ⟶ 19 ⟶ 20 ⟶ 21 ⟶ 22 ⟶ 23 ⟶ 24 ⟶ 25 ⟶ 26	2 + 12 = 14

What happens if the egg still does not break?

If the egg did not break from the 27th floor, then you would have to check from the 39th floor. And if it breaks from being dropped from the 39th floor, you would only have to check the floors from the 28th to the 38th one (11 floors). Meaning that in the worst-case scenario, we would still only have to the check 14 floors in total: 14, 27, 39, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38.

1st egg breaks a from floor…	**Test the 2nd egg from floor…**	**T Worst-case total drops**
39	28 ⟶ 29 ⟶ 30 ⟶ 31 ⟶ 32 ⟶ 33 ⟶ 34 ⟶ 35 ⟶ 36 ⟶ 37 ⟶ 38	3 + 11 = 14

Can you spot the pattern yet?

Using the same logic, you should then move up to check the 50th floor and repeat the process. Then the 60th floor, the 69th floor, the 77th floor, the 84th floor, the 90th floor, the 99th floor and finally the 100th floor.

Putting this all together gives us:

1st egg breaks from floor…	**Test the 2nd egg from floor…**	**Worst-case total drops**
14	1 ⟶ 2 ⟶ 3 ⟶ 4 ⟶ 5 ⟶ 6 ⟶ 7 ⟶ 8 ⟶ 9 ⟶ 10 ⟶ 11 ⟶ 12 ⟶ 13	1 + 13 = 14
27	15 ⟶ 16 ⟶ 17 ⟶ 18 ⟶ 19 ⟶ 20 ⟶ 21 ⟶ 22 ⟶ 23 ⟶ 24 ⟶ 25 ⟶ 26	2 + 12 = 14
39	28 ⟶ 29 ⟶ 30 ⟶ 31 ⟶ 32 ⟶ 33 ⟶ 34 ⟶ 35 ⟶ 36 ⟶ 37 ⟶ 38	3 + 11 = 14
50	40 ⟶ 41 ⟶ 42 ⟶ 43 ⟶ 44 ⟶ 45 ⟶ 46 ⟶ 47 ⟶ 48 ⟶ 49	4 + 10 = 14
60	51 ⟶ 52 ⟶ 53 ⟶ 54 ⟶ 55 ⟶ 56 ⟶ 57 ⟶ 58 ⟶ 59	5 + 9 = 14
69	61 ⟶ 62 ⟶ 63 ⟶ 64 ⟶ 65 ⟶ 66 ⟶ 67 ⟶ 68	6 + 8 = 14
77	70 ⟶ 71 ⟶ 72 ⟶ 73 ⟶ 74 ⟶ 75 ⟶ 76	7 + 7 = 14
84	78 ⟶ 79 ⟶ 80 ⟶ 81 ⟶ 82 ⟶ 83	8 + 6 = 14
90	85 ⟶ 86 ⟶ 87 ⟶ 88 ⟶ 89	9 + 5 = 14
95	91 ⟶ 92 ⟶ 93 ⟶ 94	10 + 4 = 14
99	96 ⟶ 97 ⟶ 98	11 + 3 = 14
100		= 12

Using this strategy, you would be able to cover all the floors and the number of attempts would never be bigger than 14!

Counting Chickens

Introduction

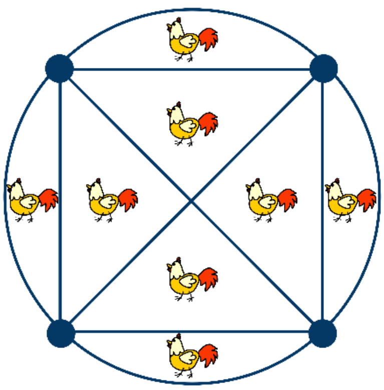
This activity looks at the interesting concept of having a circle with n points on its perimeter and what happens when you joint those points up.

Three examples are given on the question sheet so students can see how to visualize the problems. The activity questions rely on students being able to develop on these primary examples to solve more complicated cases. The extension looks at spotting a pattern and developing a rule.

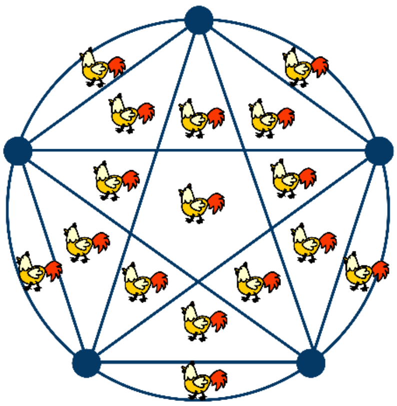
It is advisable that students try drawing out each case. It should be noted that the use of colours can be a way for students to picture the different regions more easily.

Solution

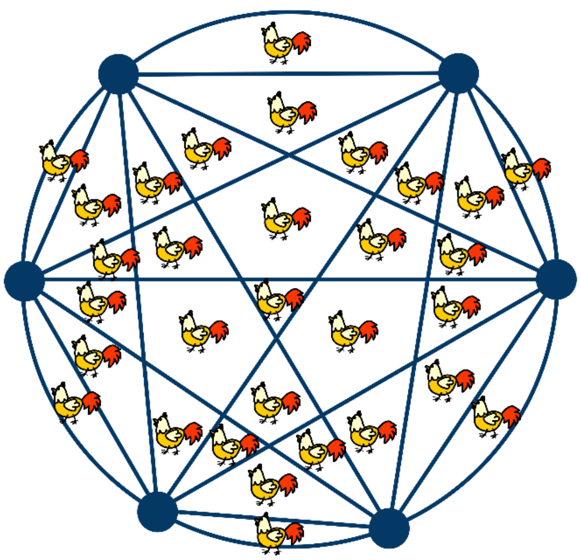
1) With 4 fence posts the farmer can keep 8 chickens.



2) With 5 fence posts the farmer can keep 16 chickens.



3) With 6 fence posts the farmer can keep 31 chickens.



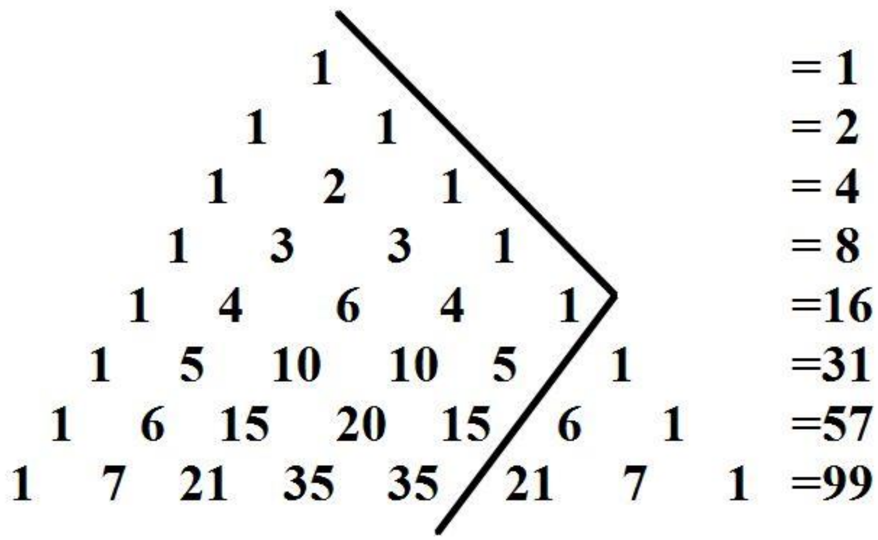
Extension

Is there a pattern between the number of fence posts and the maximum number of chickens that can be kept?

Hint: Fill out this table (some of it has been filled out for you)

Number of fence posts	Number of chickens
1	1
2	2
3	4
4	?
5	?
6	?
7	?
8	?

Using Pascal's triangle, what will happen if you add the numbers to the left of the line in each row? Can you spot the pattern now? Can you explain this strange pattern?



Pell Numbers

Introduction

This activity starts with trial and error to find some values of **a** and **b** that work. Set out any correct answers that students get in a table like this, putting the smallest solutions at the top.

b	a	Answer

Students will hopefully spot some patterns appearing. If you have access to a computer or tablet, you could try to generate more solutions on a spreadsheet such as GeoGebra or Excel.

Solution

b	a	Answer
0	1	-1
1	1	1
2	3	-1
5	7	1
12	17	-1
29	41	1
…	…	

You can find the next row for **b** by adding the previous answers for **a** and **b** together.

You can find the next for **a** by adding the previous answer for **a** and the new value for **b**.

e.g. **29 = 12+17** and **41=29+17**

You can also find new values for **a** and **b** by adding together twice the previous term and the term before that.

e.g. **29=2x12+5** and **41 = 2x17+7**

Mathematically, for the **b** column, this is written as $\boldsymbol{b_n = 2b_{n-1}+b_{n-2}}$ where $\boldsymbol{b_0 = 0}$ **and** $\boldsymbol{b_{1=}1.}$

$\boldsymbol{\frac{a}{b}}$ for each row gives an approximation to $\sqrt{\mathbf{2}}$. The approximation gets better as you go down the table. But it will never be perfect no matter how long you do this for!

Extension

Try doing the activity again but starting with $\sqrt{\boldsymbol{3}} = \boldsymbol{\frac{a}{b}}$. More info can be found at https://en.wikipedia.org/wiki/Pell_number. Thanks to Kevin Buzzard from Imperial College for introducing SAMI to the activity.

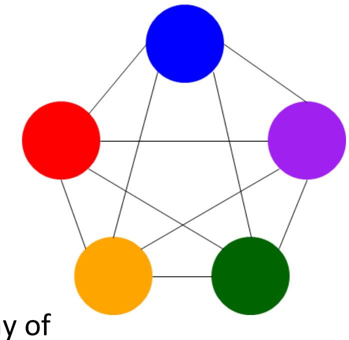
Balls and Books

Introduction

This is a follow-up to "Handshakes". The idea is to show different ways of thinking about a problem, and realize how seemingly different problems can be related to each other. You should encourage students to be systematic and decide on a precise and ordered way to describe what has been selected.

Solution

1. There are different ways to approach the first problem. One is trying to list all the possible combinations. A good question to ask is: **how to write down a single combination?** One could write down two letters, the letters being the initials of the colors (e.g. BY for "blue and yellow"). Or assign numbers to the colors and write down these.



Another way is to draw each of the balls on paper, and connect possible pairs (this way it is clear that the problem is the same as the handshake problem).

Either way, one obtains **<u>10 different combinations</u>** for 5 balls.
(12, 13, 14, 15, 23, 24, 25, 34, 35, 45).

To solve the problem for more balls, one can ask the following question: **Which new combinations arise when adding another ball?** The new ball can be combined with any of the 5 previous balls, giving 10+5=15 solutions for 6 balls. Similarly, we get 15+6=21 solutions for 7 balls.

A different way of solving this problem is realizing that each of the 5 balls can be combined with each of the other 4 balls, giving 5*4=20 combinations. However, doing this we get each combination twice (e.g. we get 12 and 21, which are the same). So we need to divide the answer by 2 to get 20/2=10.

2. Again, ask the two questions:

How to write down a single combination? The most obvious way is to write down a string of letters representing the color, e.g. (BGGBG).

A different way is to write down the two positions where the 2 blue books are. In the example, they would be in positions 1 and 4, so write 14. This is less obvious and it's fine if the students don't find it. If they do however, they will see that this problem is the same as the first one, and so there are **10 possibilities**!

Which new combinations arise when adding a 4th green book? There are two kinds of line-ups to consider: Those where the last book is green, and where it is red. If the last book is green, then removing it gives us a line-up for 2 red and 3 green books. We already know there are 10 of them! If the last book is red, then there are 5 different possible positions for the other red book. Thus there are 10+5 combinations for 2 red and 4 green books. Similarly, 21 for 2 red and 5 green books.

Extension

Consider the first problem again, but this time taking 3 balls out each time.
Try to decide a general rule or formula for any of the problems (e.g. picking 2 balls from N in a bag)

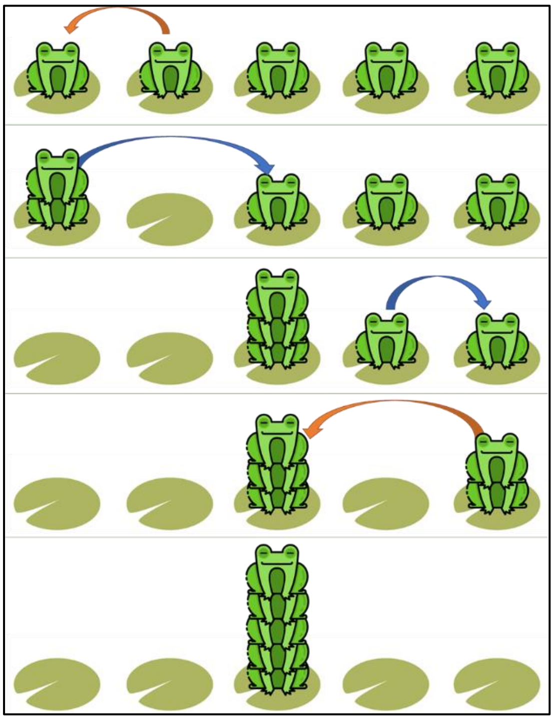
Frog Party

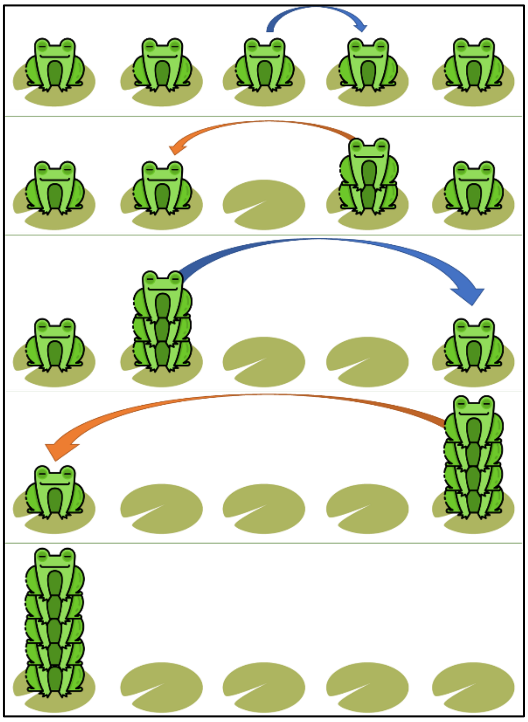
Introduction

This is a really fun problem, and can be used to build many much harder problems. It is important when starting to problem to make sure the rules are clear. You might want to do this by showing examples of moves which are and are not allowed, using coins on top of pieces of paper or similar.

Solution

There are many ways to solve this, and it is possible to have a party on every single lily pad. Two examples are given below, however you should ask students to also share their solutions





Extension

There are lots of ways you can make this problem more challenging and interesting, which include:

- Add more lily pads, is it still possible with 6, 10 or N?
- Choose one of the frogs to be the Queen frog, who must be at the top of the party
- Swap one of the frogs for a lazy toad, who refuses to move. Is it still possible?

Note, the strategy used in the second solution of working from the inside out also works for any number of lily pads!

Monkey Business

Introduction

This is another problem where the solution seems impossible because there are too many things to consider (1000 monkeys!), but again becomes achievable once simplified and able to see patterns. To start the problem you should ensure students understand the rules, perhaps imitating with 4 students and the same number of coins or cards that can be flipped to face up or down.

The first student flip all cards up (u):	[u, u, u, u]
The second student flip the evens down (d):	[u, d, u, d]
The third student would flip card 3:	[u, d, d, d]
The fourth student would flip card 4:	[u, d, d, u]

Solution

1. For the first part of the problem you only need to consider 10 monkeys (as the 11th monkey and later will not press any of the first 10 switches). So which monkeys will press switch number 10? Well, as 10 is divisible by 1, 2, 5 and 10 there are exactly 4 monkeys which will press the switch

- Monkey 1 turns it on
- Monkey 2 turns it off
- Monkey 5 turns it back on
- Monkey 10 turns it back off

So switch 10 will be **off**.

2. The next part is more challenging, but again relies of thinking about how many monkeys will press each switch.

Switch 10 was off because every time one monkey turned it on another turned it off. We were able to split the number 10 into factor pairs (1x10, 2x5), which will always result in the light being turned off.

Most numbers will be pressed by an even number of monkeys, e.g. 24 has factors (1x24, 2x12, 3x8, 4x6), so monkeys 1, 2, 3, 4, 6, 8, 12, 24 will press the switch, and as this is an even number the light will be off.

Square numbers are the only ones which do not, e.g. 16 has factors (1x16, 2x8, **4x4**). Monkeys 1, 2, **4**, 8, 16 will press the switch, and as this is an odd number the light will stay on (monkey 4 won't press it twice!).

So we just need to work out how many square numbers there are between 1 and 1000?

If we think about the larges square number less than 1000, **30x30=900, 31x31=961**, **32x32=1024**

So there will be **31 switches left on**! (switch numbers 1, 4, 9, 16, 25, 36, … , 900, 961)

Extension

How many lights would stay on if only the even numbered monkeys decided to take their turn pressing the switches?

Locks and Keys

Introduction

This problem is not specified very precisely, and in this case it is intentional. The aim of this problem is to work with assumptions. The problem has different solutions under different assumptions, and it might be easy or even impossible depending on the assumptions. For example, if we assume that the messenger can break the box or the lock, then the problem is impossible (and somewhat pointless).

There is no right or wrong solution because there is no right or wrong set of assumptions. However, if we specify our assumptions clearly, then for the given set of assumptions any solution is either right or wrong. The facilitator should encourage students to decide on what should be allowed or not so that the problem is not solved too easily or made impossible.

This activity involves lots of discussion between groups. It could help to have paper cutouts of locks, keys and boxes so students can demonstrate rather than only explain. You should encourage students to demonstrate their solutions, and to challenge each other to see if they can find a way to intercept.

Solution

There's are lots of different strategies that could be tested and argued, here we will only outline a few possibilities, it is up to the students and facilitator to decide if their own solutions work or not.

We will start with a few assumptions:

1. The messenger cannot break the box or the lock.
2. The messenger cannot make copies of keys.

Case 1: Alice locks the box and sends it to Bob. Bob sends the messenger back to Alice. Alice now gives the messenger the key to send it to Bob. The messenger gives the key to Bob and Bob unlocks the box.

Intercept: The messenger only pretends to deliver the box to Bob, but instead keeps it. He then returns it back to Alice and obtains the key to unlock the box and read the message.

Case 2: Alice sends the key using a different messenger than for the box.

Intercept: The messengers collaborate. You would need to decide in the rules whether this would be possible.

Case 3: Alice locks the box with her lock and sends it to Bob. Bob attaches his own lock and sends it back. Alice takes her lock off the box and sends it back again. Finally, Bob takes his own lock off to read the message.

Intercept: In this case there would be no way to intercept the message without breaking the box or copying keys!

Extension

Bob opens the box and see it is instructions to play a game. To start the game Alice and Bob will flip a coin to decide who goes first, however they will still need the messenger to communicate the outcome. Who should flip the coin? Who should choose heads or tails? How can they communicate in a way so that neither person can cheat?

Paths

Introduction. This is a follow-up to “Balls and Books”. It is quite challenging as it introduces Pascal's triangle and using Pascal's triangle to solve other combinatorial problems. Students will very likely need some guidance (in the form of nudges and simpler intermediate tasks) to solve the first problem.

Solution

1. Problem 1 might look scary at first. Trying to list all possible path will be hopeless, as we'll see there are too many. **Encourage the students to be systematic and solve smaller problems to points that are closer to A first!** A good task to **write, for each point on the grid, the number of paths to that point.** For some points this will be easy, for example for the points in the first row or first column there is only 1 path. For the points in the second row or column this is a little harder, but still possible. For a point in the second row, you can only go down once. If the point is in the third column for example, then there are 3 choices where to go down: Either in the first, second or third column. Thus, in the second row the numbers will be: 1, 2, 3, 4, 5, 6, 7.

From here on, it gets much harder. One question to lead onto the right track is: **Can you get the number of paths from A to B, if you already know the number of paths from A to all the points closer to B?** The key realization is that if you want to know the number of paths to a point X, and you know the number of paths for the point above X and for the point to the left of X, then the number of paths to X is simply the sum of number of paths for the point above X and the number of paths for the point to the left of X! This way, it is easy to fill in the grid by simply adding up numbers. It will look like this. **From A to C there are 42 paths, from A to B there are 210 paths**.

1	1	1	1	1	1	1
1	2	3	4	5	6	7
1	3	6	10	15	21	28
1	4	10	20	35	56	84
1	5	15	35	70	126	210

2. Some students might have already solved problem 2 when trying the first problem. Considering one can only go **D**own and **R**ight, the natural thing to do is simply writing down a sequence of R's and D's. The example path would be RDRRRDRRDD. Writing it down as numbers is slightly less straight-forward. One could write down the columns where the path goes down, however this will not help with solving the next problem. Instead, we can **write down only which path segments are going down**. In our example, the 2nd, 6th, 9th and 10th segment goes down, so we'd write 2, 6, 9, 10.

Extension

A) If students have already done the “Balls and Books” problem you could ask to try and use the same methods to solve the problem of arranging 4 blue and 6 green books on a shelf
Note: Instead of using RDRRRD, you are now using book colours BGBBBG – see the similarity! The answer will be 210 again

B) Attempt the extension activity from “Balls and Books”, to calculate the number of ways of taking 3 balls out of a bag with 5 different colours.
Note: If you numbered the balls you are considering how many ways there are of picking 3 numbers from 5 (e.g. 123, 145, 234 etc.). This looks very similar to the situation above if had 5 columns (representing the colours), and could take 3 down steps (picking a colour). The solution in this case will be 35!

Scales and Weights

Introduction

The statement is not 100% precise, so: You want to put all the flour in one bag and weigh it, not splitting it into smaller bags for weighing. The weights to measure are whole numbers only.

Solution

This problem has two different versions depending on the assumptions you're using. Both are interesting, and it is worth **encouraging students to try both.**

- The first version is probably slightly easier, and assumes that you can only put weighing stones on the side of the scale.
- In the second version, you can put weighing stones on either side, giving more options.

Encourage students to try for much smaller numbers than 40 first. For example, if we only want to measure weights up to 7kg, then weighing stones of 1 kg and 2 kg and 4kg are sufficient (check that you can measure 1,2,3,4,5,6,7kg with these!).

If we could put weights on both sides we could come up with more interesting solutions, for example if we had 1kg, 3kg and 6kg weights we could weight 1,2,3,4,5,6,7,8,9, and 10!

E.g. If we have 8kg of flour, we could add the 1kg weight to it and it would balance against the 6kg and 3kg. See if you can make all the other numbers!

The best strategy for the first problem is to start from 1kg, and every time there is a number you can't make add another stone. Quickly you see that if you have **1kg, 2kg, 4kg, 8kg, 16kg**, you can measure every number up to 31kg. You will therefore need a 5th stone to reach 40kg. You could make this a 9kg stone, however it would be better if you added a **32kg** stone and now can measure everything up to 63kg.

For the second problem you not only want to consider the total when stones are added together, but the difference when they are subtracted (because you can have stones on either side). You want to give as many different numbers as possible using as few stones as possible.

To start we will either need a 1kg stone, or 2 stones with a 1kg difference. In the example above we started with a 1kg stone but what if we started with **1kg** and **3kg**? We can now measure 1,2,3,4.
To get 5kg we can pick the largest number which will balance against the 5kg flour + 1kg + 3kg stones, so we should pick **9kg**. This now lets us make every number up to 9+3+1 = 13kg. To get 14kg we simply add a **27kg** stone, because 27kg – 9kg – 3kg – 1kg = 14. With the 27kg stone we can now measure everything up to 40Kg, so **it is possible with only these 4 stones!**

Extension

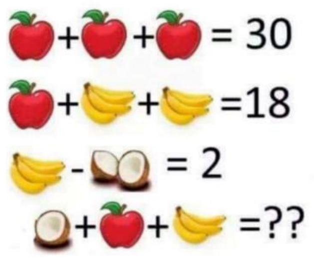
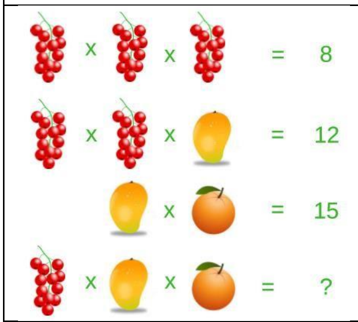
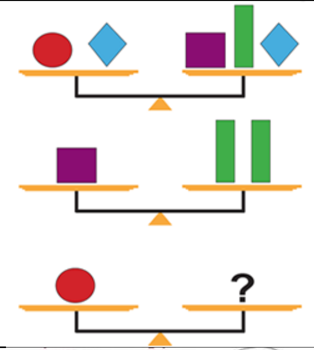
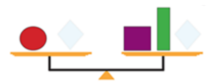
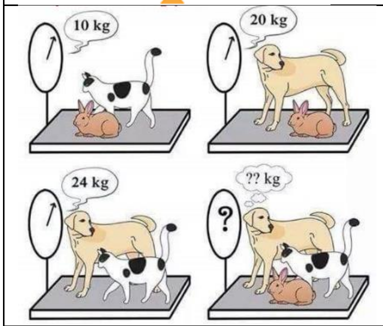
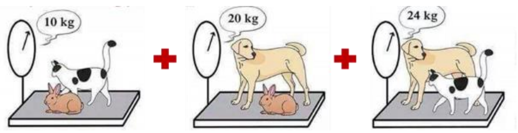
Continue this process to see which stones you would use up to 1000kg. Is there a formula you could use to calculate?

Picture Puzzles I

Introduction

These problems can all be solved in different ways, but it is recommended you encourage students to use logic and reasoning skills instead of guessing numbers. Usually each line gives you a piece of information that you can use in the next, so you should think carefully about each piece of information before moving onto the next. The problems also get harder as you go, so encourage students to start with the top-left puzzle first.

Solution

	3 apples = 30, so **1 apple = 10** 10 + 2 bananas = 18, so **1 banana = 4** 1 banana – 1 coconut = 2, so **1 coconut = 2** Altogether you get 2 + 10 + 4 = **16**
	2 x 2 x 2 = 8 2 x 2 x 3 = 12 3 x 5 = 15 2 x 3 x 5 = **30**
	Remove the diamond from both sides and the balance will still be level, so the circle is the same as one square and one rectangle  The second line tells us that the square is the same as 2 rectangles so we could replace in the square in the image above 
	Add all the animals in the 3 known scales together  This will give you **2 cats, 2 dogs, and 2 rabbits** with a total weight **= 54kg** Therefore **1 cat, 1 dog and 1 rabbit** will have a total weight **27kg** (you don't need to work out all the individual weights!)

Extension

Ask students to try and make their own picture puzzles!

Make Many

Introduction

This type of puzzle comes from a TV show. The first task should help students practice techniques to use in Task 2. Students might be surprised at how many numbers you can make using only 100, 2 and 3. The full version of the game is presented in the game 'Countdown' (page 19)

Students could start on their own and then share their answers to see if as I group they can find them all. Don't tell them there are 19 until they get there!

Solution

Task 1

100, 101, 102, 103, 105, 106, 150, 194, 197, 200, 203, 206, 294, 298, 300, 302, 306, 500, 600
Some of the tricky ones are, e.g. 194 = $(100 - 3) \times 2$

It shows how useful brackets are!

Task 2

Here is one way to solve each problem. Students may come up with other correct solutions – just check they have only used each number once!

$635 = 6 \ \times 100 + 7 \times 5$

$667 = 6 \times 100 + 50 + 5 + 5 + 7$

$665 = 7 \times (100 - 5)$

564 =$6 \times (100 - 7 + \frac{5}{5})$

785 = $7 \times 100 + 50 + 5 \times 6 + 5$

202 = $= (7 - 5) \times (100 + \frac{5}{5})$

420= $6 \times 7 \times 100 \div (5 + 5)$

$419 = (7 \times 6 \times 50 - 5) \div 5$

Extension

Have a competition following the game rules on page 19.

Secret Santa

Introduction

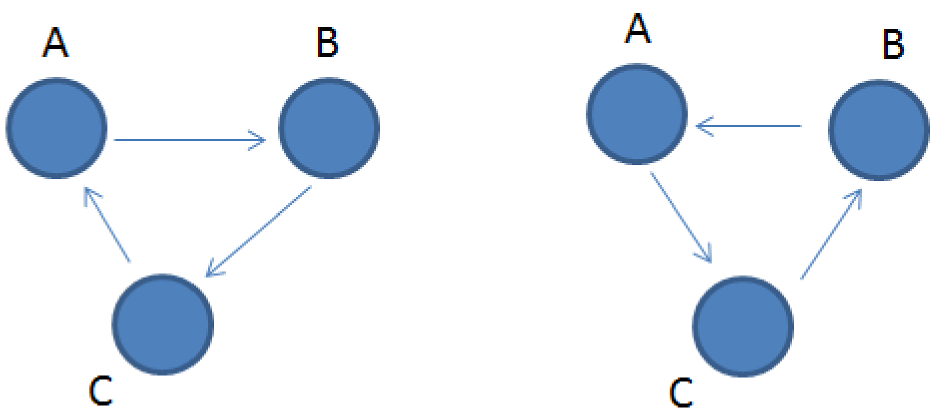
In this problem, we are trying to count the number of possible ways to have a secret Santa. The idea behind secret Santa is that the names of all the people involved are put in a hat and each person has to pick one name from the hat, not including your name.

Each stage gets progressively harder in the hope that the students start to understand the general rules behind the way to find the number of possible scenarios.

Firstly, let the students think about ways to represent this problem. The students could even arrange themselves in groups to reenact different scenarios and count the number of different ways of having a secret Santa.

Solution

<u>3 people</u>

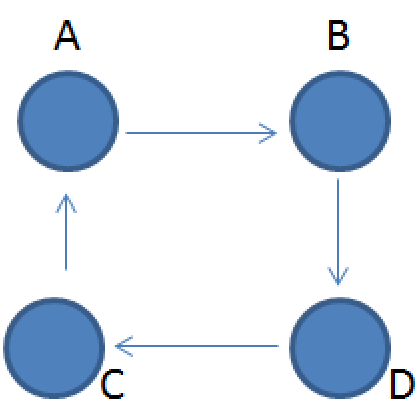


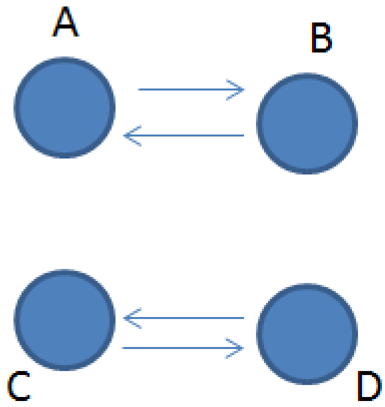
In this problem, you might have noticed that if A could either give to B or C, thus there are the 2 circular solutions above.

3 people, 2 ways

<u>4 people</u>

For 4 people, if you looked at the possibilities using networks, you might have realised that the network can be decomposed into 2 networks with 2 people in them:

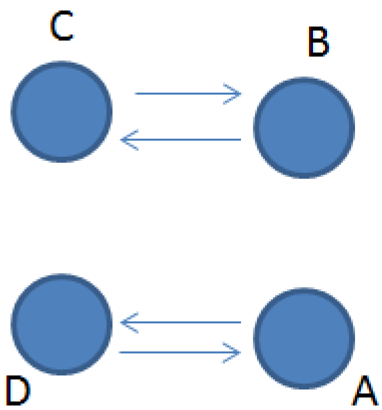


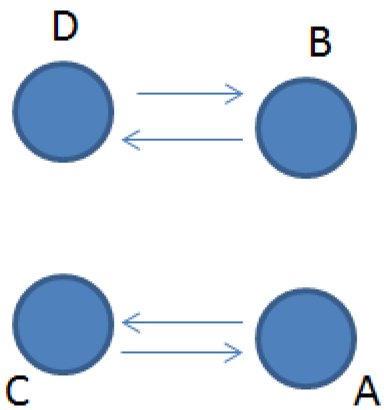


The problem is therefore how many different ways do you have of having everyone in a same network and having 2 smaller networks with 2 people each?

The answer is that for the larger network, there are 3! (= $3 \times 2 \times 1$) ways of doing it as there are 3! different ways of changing the order of neighbouring nodes. You could write all these options as a list that you can imagine into a circle: ABCD, ABDC, ACBD,ACDB, ADBC,ADCB.

For the second question, we can see that there are 3 possibilities, which are the one above and the two below:





Therefore, for 4 people, you have 6+3 = 9

4 people, 9 ways

<u>5 people</u>

There are two different options. A circle of 5, which could be done in 4! = $4 \times 3 \times 2 \times 1 = 24$ ways.

A circle of 3 and a pair. The 3 could be chosen in 10 different ways and then there are two ways round the circle so that is 20.

5 people, 44 ways

Extension

How about 6 people? 7? There are some properties of the sequence 2, 9, 44, etc. here https://oeis.org/A000166

Computer Activities

African Maths Initiative

Logo Challenge 1

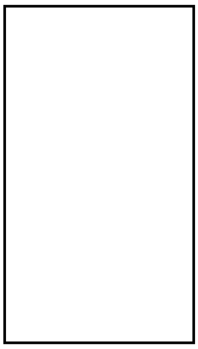
For this problem you will need MSW logo installed on the computer:
http://mswlogo.en.lo4d.com/download/mirror-hs1

Introduction

Logo uses basic commands to draw shapes. The most common are:

Fd 10 – move forward 10 steps (you can change 10 for any number)
Bk 20 – move backwards 20 steps
Rt 90 – turn right 90 degrees (you can change 90 for any number)
Lt 200 – turn left 200 degrees

For example to draw the following shape you could give instructions:



fd 100
rt 90
fd 50
rt 90
fd 100
rt 90
fd 50

You can also make things faster by using the *repeat* command:

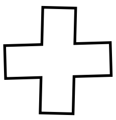
Draw the same shape by inputting

Repeat 2 [fd 100 rt 90 fd 50 rt 90]

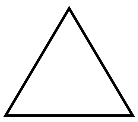
This will repeat 2 times the instruction that has been written between the square brackets [ ]
You can find the [ ] keys on the keyboard

More resources on using logo can also be found here: http://www.softronix.com/logo.html

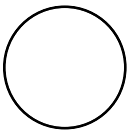
Solutions



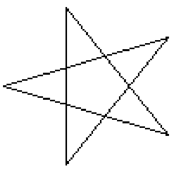
Repeat 4 [fd 50 rt 90 fd 50 rt 90 fd 50 lt 90]



Repeat 3 [fd 100 rt 120]



Repeat 360 [fd 1 rt 1]



Repeat 5 [fd 100 rt 144]

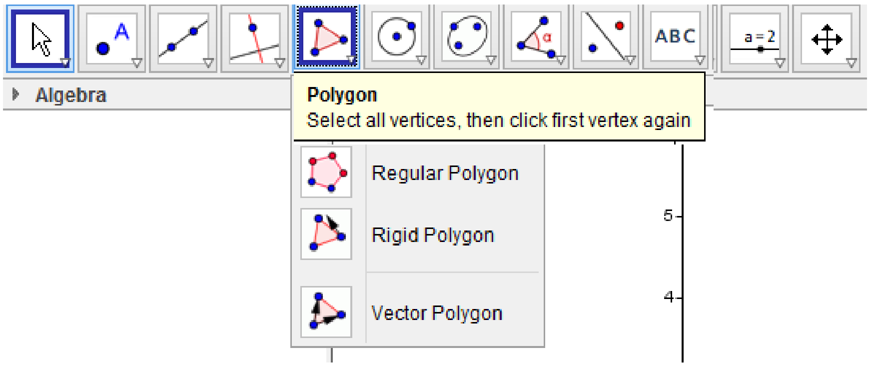
Extension
Draw other shapes on the board for learners to try and construct

GeoGebra Challenge 1

For this problem you will need GeoGebra installed on the computer: http://www.geogebra.org/download

Introduction

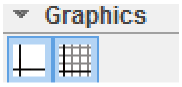
GeoGebra is a powerful tool for geometry, algebra and statistics. When you start GeoGebra you will see a list of tools you can use:

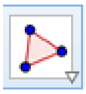


You can use the dropdown arrows to gain more tools. Hovering the mouse over a tool provides instructions for use.

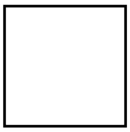
Solutions

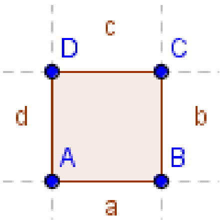
There are many ways to make each shape, here are a few example solutions.
Hint, to show the grid you click on the arrow next to the word graphics



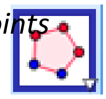


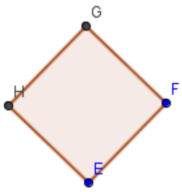
1a. Use the polygon tool and click on coordinates that make a square

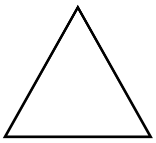




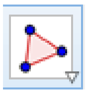
1b. Use the regular polygon tool and click any two points

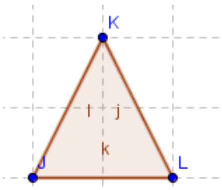




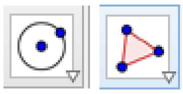


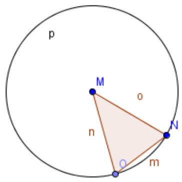
2a. Use the polygon tool and click on coordinates that make an isosceles triangle



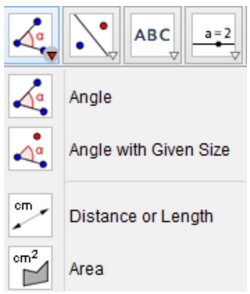


2b. Draw a circle. Connect two points on the circumference and the centre to form an isosceles triangle





Proof: Use tools such as *angle* or *length* tools to verify that the shapes are as described. Alternatively pupils may argue with ideas such as symmetry, Pythagoras, circles etc.



Extension: Try to create other shapes such as a trapezium, rhombus or ellipse

Scratch Challenge 1

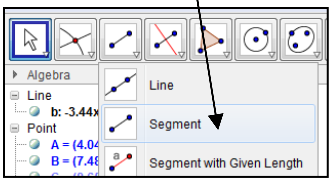
See scratch tutorial document

GeoGebra Challenge 2

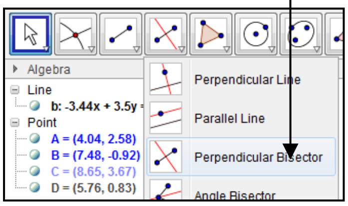
For this problem you will need GeoGebra installed on the computers: http://www.geogebra.org/download
See GeoGebra Challenge 1 for an introduction to GeoGebra.

Solutions

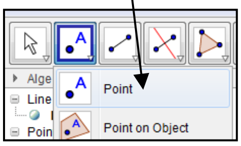
1. Use the *segment* tool then click two points A and B on the graphics view.

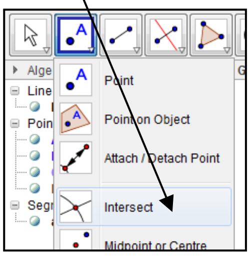


2. Use the *perpendicular bisector* tool and then click on the segment. (Make sure no one used the Perpendicular line instead!)



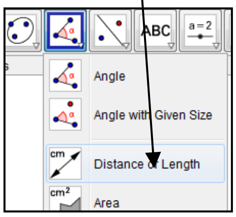
3. Use *point* tool for C and the *intersect* tool for D and click on the intersection point.

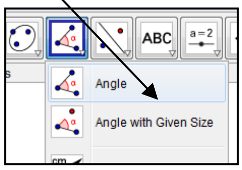




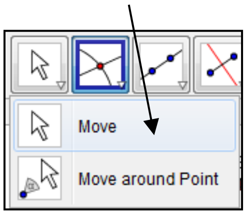
4. The perpendicular bisector cuts the segment into two equal pieces and makes an angle of 90° with the segment.

To use *length*, click the two end points of a segment. To use *angle*, click the three points on the lines in anti-clockwise order. You can also discuss what happens if you click points in clockwise order.





Use the *move* tool in the first box to move points.



If you move A or B, the distances change, but the angles stay the same. The perpendicular bisector also changes, but the lengths AD and DB are always equal.

5. Use *segment* again to construct AC and BC. There are 3 triangles, a big isosceles triangle and two smaller right angled triangles inside the big triangle.

6. AC and BC are the same. This is because triangles ADC and BDC are **congruent**. This means corresponding sides and angles are equal. We can prove this using the Side Angle Side (SAS) theorem.

AD and DB are equal because the perpendicular bisector cuts the line into equal pieces. The angles ADC and BDC are equal and both 90° because the perpendicular bisector makes a perpendicular line to AB, and finally the side DC is shared by both triangles so they are equal. Hence, we have two corresponding sides and their enclosed angle being the same in both triangles, so they are congruent by the SAS theorem. (If students are not familiar with this theorem you can describe it to them first.)

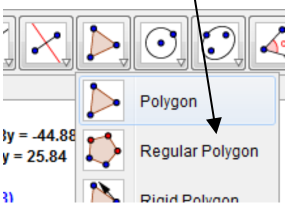
Extension: Move on to GeoGebra Challenge 3 to do more activities with perpendicular bisectors

GeoGebra Challenge 3

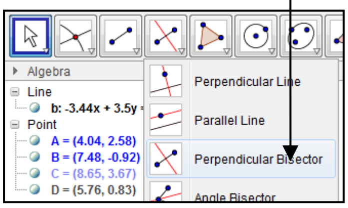
For this problem you will need GeoGebra installed on the computers: http://www.geogebra.org/download
See GeoGebra Challenge 1 for an introduction to GeoGebra.

Solutions

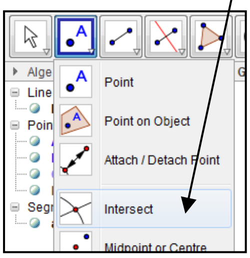
1. Use the *polygon* tool then click three points A, B and C on the graphics view then click A again.



2. Use the *perpendicular bisector* tool and then click on the segments. (Make sure no one used the Perpendicular line instead!)

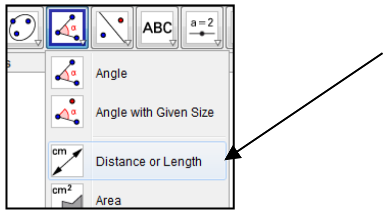


3. Use the *intersect* tool for D and click on the intersection point.



4. AD = BD. We saw this in GeoGebra Challenge 2 when we constructed the perpendicular bisector of a segment. Points on the perpendicular bisector and the same distance away from the end points.

Use *length* tool to check this by clicking the two end points of a segment.



The same is true for BD and CD for the same reason.

5. Now we know that AD = BD and BD = CD. This means AD = CD also.

6. Since D is the same distance away from A and C it must be on the perpendicular bisector of AC. If we construct the perpendicular bisector of AC it should go through D.

7. No. Use the *move* tool in the first box to move points. When all angles are less than 90° it is inside. When one angle equals 90° it is on the hypotenuse. When one angle is bigger than 90° it is outside the triangle. (You can use angle tool to check this)

8. The circle will also pass through B and C. Check this with a circle centre D through B or C. This is because AD = BD = CD so these lengths are all equal to the radius and A, B and C are on the circumference.

D is called the circumcentre of the triangle. Move the points to see how D and the circle changes.